San Diego State University

INTEGERS: HISTORY, TEXTBOOK APPROACHES, AND CHILDREN'S PRODUCTIVE MATHEMATICAL INTUITIONS

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Max (1st grade) reasons about $3 - 5 =$ _

Counterintuitive Nature of Negatives

- You saw Max reasoning about 3 – 5 = _.
- For kids like Max,
  - Numbers represent *something,*
    - and something is more than nothing,
  - Addition cannot make smaller.
  - Subtraction cannot make larger.
- This is all perfectly reasonable based on their experience. In fact, it makes a lot of sense.

Who We Are

- *Mapping Developmental Trajectories of Students' Conceptions of Integers*
- aka Project **Z**
- Our goal is to understand how students of various ages think about integers.
- In 2010, we conducted about 90 pilot interviews with K–12 students.
- In spring of 2011, we conducted 160 interviews—40 each with children in Grades 2, 4, 7, and 11.

Overview

- The Counterintuitive Nature of Negative Numbers
- Keys to the Acceptance of Negatives Historically
- Children's Productive Integer Reasoning
- Textbook Approaches
- Instructional Implications

Counterintuitive Nature of Negatives

- "Three minus five is zero because you have 3 and you can't take away 5 so take away the 3 and it leaves you with zero."
  - Sam (1st grade)
- "4 + □ = 3 is not a real problem. It's not true… Four minus 1 would equal 3."
  - Brad (1st grade)

Counterintuitive Nature of Negatives

- As it turns out, Sam is in good company:
- “I know some who cannot understand that to take four from nothing leaves nothing.”
  - Blaise Pascal

Counterintuitive Nature of Negatives

- Brad is in good company too:
- Brad said, "4 + □ = 3 is not a real problem. It's not true… Four minus 1 would equal 3."
- Diophantus claimed that the equation $4x + 20 = 4$ was "absurd": How could you *add* something to 20, and end up with 4?
- d'Alembert argued that the equation $x + 100 = 50$ was misleading and should really have been written as $100 - x = 50$.

Counterintuitive Nature of Negatives

- "Above all, [man] must reject the definition still sometimes given of the quantity $-a$, that it is less than nothing. It is astonishing that the human intellect should ever have tolerated such an absurdity as the idea of a quantity less than nothing; above all, that the notion should have outlived the belief in judicial astrology and the existence of witches, either of which is ten thousand times more possible."

Augustus De Morgan

Counterintuitive Nature of Negatives

- For these children and mathematicians alike, the idea of "a quantity less than nothing" was an oxymoron.
- Experiences with nonnegative numbers:
  - Nonnegative numbers could be associated with quantities:
    - 7 feet, 7 pounds, 7 people, etc.
  - *Greater than* and *less than* could be interpreted in the sense of magnitude, or absolute value:
    - 7 > 3.
  - The distinction between magnitude and order is obscured:
    - 7 is more than the 3, *and* 7 comes after 3.

Keys to the Acceptance of Negatives

- Negative numbers were useful *within* mathematics.
  - Negative numbers enabled mathematicians to solve algebraic equations that could not be solved otherwise.
  - The extension to a number domain that included negatives alleviated difficulties and afforded nice properties.
- Mathematicians let go of the quantitative interpretation of number.
  - Order took precedence over magnitude, so that it became sensible to say such things as $-7 < 3$.
  - The advent of abstract algebra enabled mathematicians to recognize that various number domains could exist, and mathematically different domains could have different properties.

Children’s Productive Ways of Reasoning

- We have seen many children reason productively about integers and integer arithmetic prior to formal instruction.
- Today, I’ll share the example of Roland.

Roland

- Roland was in 4th grade.
- He had been introduced to negative numbers, which he read as “minus” numbers.
- He could relate negative numbers to positive numbers pn the basis of their ordinal relationship.
- However, he had not been taught any rules for arithmetic involving negative numbers.

Roland Solves -5 + -1 = _

Roland’s Reasoning

- How did Roland think about -5 + -1? Think and talk with your neighbor.
- Next, Roland solved -5 – -3 = _
- How might he have reasoned about this problem?

Roland solves $-5 - -3 =$ _

Roland's Reasoning

- Roland successfully solved -5 – -3.
- He did this by reasoning that subtracting -3 should have the *opposite* effect as adding -3.

Children Reasoning Meaningfully

- Many of the K–4 students we have interviewed have looked like Max.
- They make sense but cannot solve tasks like 3 – 5.
- These kids operate in a "Whole Number World" in which properties such as addition cannot make smaller and subtraction cannot make larger.
- For these kids, numbers tell you "how much" or "how many."

Children Reasoning Meaningfully

- Other K–4 students look something like Roland.
- They make sense, *and* they can solve our tasks.
- These kids operate in a world of positive and negative numbers, which are related to one another by their *ordering*.
- They also leverage mathematical properties and draw logical conclusions from what they know.
- Distinguishing characteristics:
  - Exposure to negatives!
  - Order vs. magnitude

Connections

- Both mathematicians and children have made progress in reasoning about negative numbers by
  - Operating within the pure mathematical realm;
  - Comparing integers on the basis of order, rather than magnitude;
  - Leveraging logic and mathematical properties to draw conclusions about what should be true.

Implications for Instruction

- Integers are useful and can be fascinating for kids *if* instructional approaches allow room for this.
- Children could be provided opportunities to make sense of integers and integer arithmetic:
  - They can reason through integer arithmetic without being given rules;
  - The sign rules could be generalizations of their solutions.
- Children can reason productively about integers as abstract entities:
  - Sense making does not require connecting integers to contexts.

Thank You

- Thank you for your attention and participation.
- We welcome your questions and comments.