



# Children's Informal Conceptions of Integers

- Welcome!
- See the handout to
  - identify appropriate grade levels when students should be able to solve the problems;
  - predict strategies students might use;
  - rank the open number sentences from easiest to hardest (you can have “ties”).

This presentation is based upon work supported by the National Science Foundation under grant number DRL-0918780. Any opinions, findings, conclusions, and recommendations expressed in this material are those of the authors and do not necessarily reflect the views of NSF.



# Children's Informal Conceptions of Integers

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Project **Z**: Mapping Developmental Trajectories  
of Students' Conceptions of Integers



Funded by the National Science Foundation,  
DRL-0918780

# Today's Goal

To describe grades 2 and 4 students' informal conceptions of integers (addition and subtraction only) and describe categories of reasoning.

# Project Goal

To describe K–12 students' conceptions of integers and operations with integers (+ and – only) and map possible learning trajectories.

# Why Study Integers?

Keep Change Change (KCC), Gr 11 & 7,  $6 - -2 = \underline{\quad}$  &  $5 - \underline{\quad} = 8$

- Computational approaches aren't bad.
- But they can come with trade-offs.
- Although most successful high school mathematics students could correctly and accurately use rules and procedures, most could not explain why they worked.

“Above all, he [the student] must reject the definition still sometimes given of the quantity  $-a$ , that it is less than nothing. It is astonishing that the human intellect should ever have tolerated such an absurdity as the idea of a quantity less than nothing; above all, that the notion should have outlived the belief in judicial astrology and the existence of witches, either of which is ten thousand times more possible.”

--Augustus De Morgan, 1898

*On the Study and Difficulties of Mathematics*, p.

72

# Modern Day Quote

“I am always early to meetings, although sometimes I am early a negative number of minutes.”

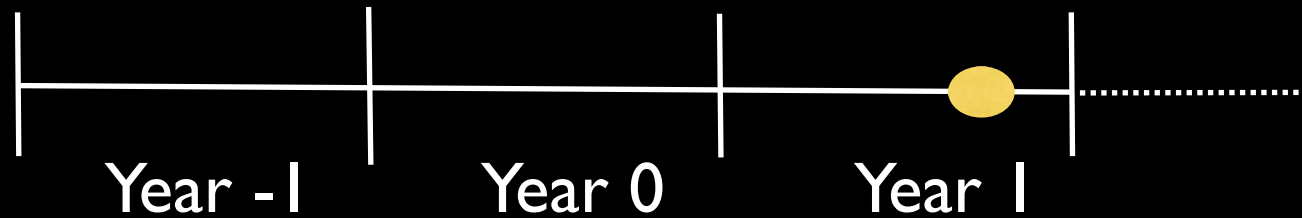
-Anonymous Zster

# Outline of Talk

- Project overview
- Negative-number challenges
- Historical perspective
- Ways of Reasoning
- Initial trends across grade bands
- Discussion/questions



# Project Overview



- Developed interview for grades K–12 and piloted more than 90 interviews
- Conducted 160 interviews--cross-sectional design
  - 40 from each of Grades 2, 4, 7, and 11 across 11 ethnically diverse school sites w/ varying API scores.
  - Problem-solving interviews lasted approximately 1.5 hours
  - Open number sentences, compare problems, story problems (majority of questions were open number)
- Beginning to code and analyze data

# Why Focus on Students' Mathematical Thinking?

- There is a large literature base surrounding students' mathematical thinking and ways of reasoning (e.g., Cognitively Guided Instruction; Karen Fuson and colleagues' work; and Purdue's Problem-Centered Mathematics Project).
- Children's mathematical thinking should inform instructional decisions as teachers use this information to productively support and extend children's reasoning (Black & Wiliam, 1998; Fennema et al., 1996; Vygotsky, 1998; Wiliam et al., 2004).

# Why Focus on Students' Mathematical Thinking?

- Children often approach problems and reason about them differently than adults do.
- Consider the problem  $-2 + \underline{\quad} = 4$ . Solve it two ways.
- Liberty is a 2nd grader who had heard of negative numbers and could add and subtract integers using counting strategies. Liberty answered 7 when solving this problem. How might she have arrived at that answer?

# Students with negative integers

- Order-based reasoning
  - Count by 1s, Liberty, Grade 2,  $-2 + \square = 4$

# What we know so far ...

- Young children have ideas about negative numbers that can be leveraged.
- Different ways of reasoning have both limitations and affordances.
- Understanding integers is complex! One way of reasoning is likely insufficient for making sense of integer operations.
- Similar to the historical development of integers, students, too, can use fundamental mathematical principles and formalisms to reason about negative numbers.

What is so hard  
about integers  
anyway?

# The Un-natural Nature of $\mathbb{Z}$

- The existence of quantities less than nothing.
- Removing something from nothing or more than you have.
- Counterintuitive situations involving routine interpretations of addition and subtraction.

# Quantities Less than Nothing?

- “A number tells you, like, how much of something it is. Negative numbers aren’t really numbers; they just act like numbers ... I mean there is no negative 1 cube” (Rosie, 2nd grade).



# Taking away more than you have?

- Andrew, Gr 2.  $3 - 5 = \square$

# Something From Nothing?

- “Three minus 5 doesn’t make sense because 3 is LESS than 5.” (Niki, Gr I)
- “You may put a mark before 1, which it will obey: It submits to be taken away from another number greater than itself, but to attempt to take it away from the number less than itself is ridiculous. Yet this is attempted by algebraists who talk of numbers less than nothing ...”

--William Frend, *The Principles of Algebra*, 1796

# Something From Nothing?

- Seth, Gr 1.  $3 - 5 = \square$

# Something From Nothing?

- Seth keeps good company.

“I know people who cannot understand that when you subtract 4 from zero what is left is zero.”

--Blaise Pascal, *Pensées*, 17th century

$$0 - 4 = 0 \quad ??$$

$$3 - 5 = \square$$

	Gr 2 ( $n = 40$ )	Gr 4 ( $n = 40$ )
-2		
Not Possible or Not Sure		
0		
2		

$$3 - 5 = \square$$

	Gr 2 ( $n = 40$ )	Gr 4 ( $n = 40$ )
-2		
Not Possible or Not Sure	32.5%	17.5%
0		
2		

$$3 - 5 = \square$$

	Gr 2 ( $n = 40$ )	Gr 4 ( $n = 40$ )
-2		
Not Possible or Not Sure	32.5%	17.5%
0	12.5%	2.5%
2		

$$3 - 5 = \square$$

	Gr 2 ( $n = 40$ )	Gr 4 ( $n = 40$ )
-2	25%	32.5%
Not Possible or Not Sure	32.5%	17.5%
0	12.5%	2.5%
2	25%	45%



# Addition Makes Smaller?

- No way!
  - Little Diophantus, 2nd grade,

$$6 + \square = 4$$

# Addition Cannot Make Smaller

$4x + \blacksquare = 4$  is “absurd” because the four units as the result of the summation “ought to be some number greater than 20.” (Diophantus, Grade ?, 3rd century)

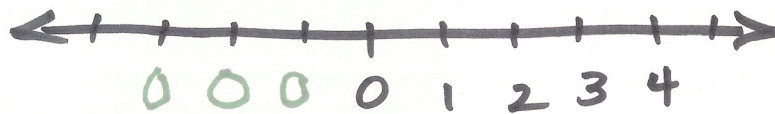
# Lessons From History

- For these children and mathematicians alike, numbers “less than nothing” were a paradox.
- Mathematical formalisms helped to broaden mathematicians’ ideas about number and encouraged the acceptance (finally!) of negative numbers. Why?
  - Negative numbers enabled mathematicians to solve algebraic equations that could not be solved otherwise.
  - The advent of abstract algebra enabled mathematicians to recognize that various number domains were possible, that including negatives afforded nice properties (e.g., inverses), and different domains could have different properties.

# Outline of Talk

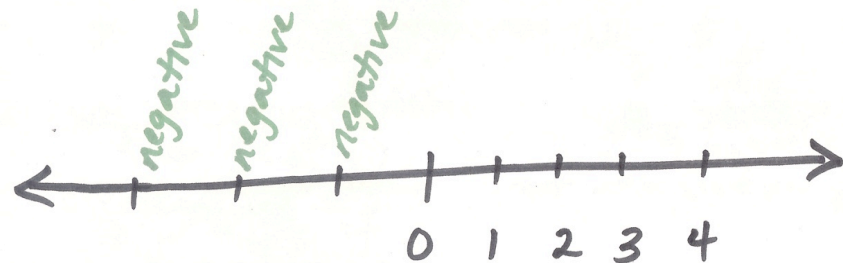
- Project overview
- Negative-number challenges
- Historical perspective
- ● Ways of Reasoning
- Initial trends across grade bands
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# Ordering and Comparing Numbers

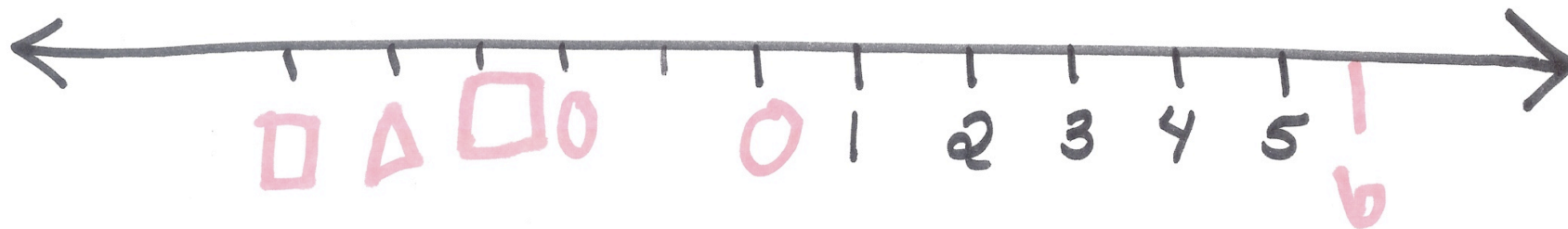


Bradlee, 1<sup>st</sup> grade  
Jan 2010

Don't forget the negative hallway, some kids wanting to call everything "zero" or "negative"

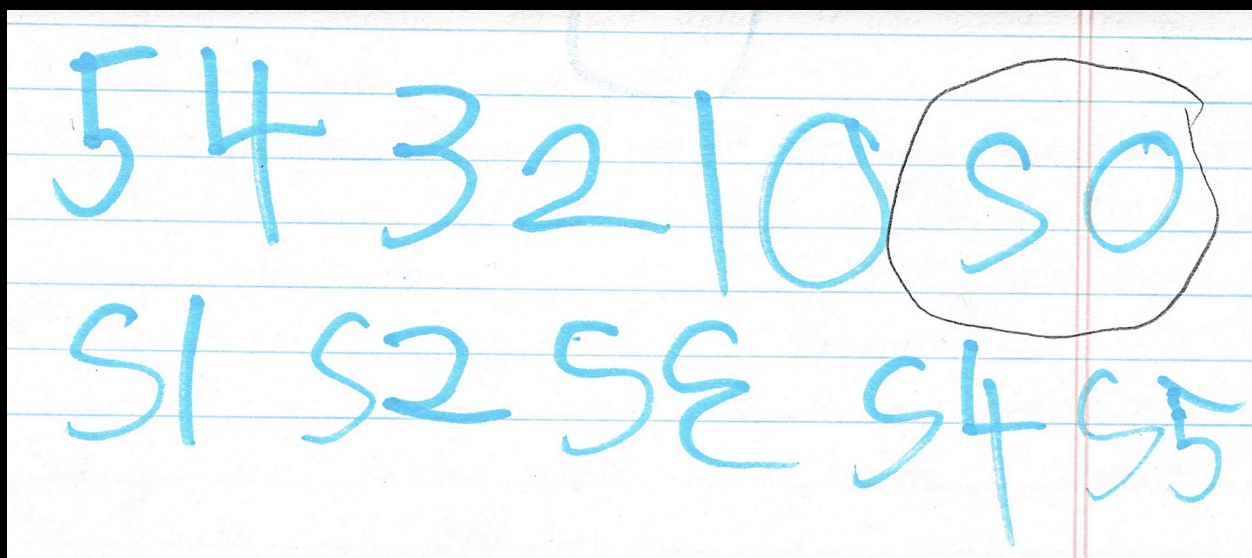


Teddy, Grade 1



Seth, 1<sup>st</sup> grade

Mar 2010



Niki's representation of the "somethings"

# Students' Integer Reasoning

	Negative integers <b>not in</b> numeric domain	Negative integers <b>in</b> numeric domain
2nd grade	27	13
4th grade	13	27
7th grade	0	40
11th grade	0	40
	40	120

# Students With Negative Integers

- Children who have negative integers in their conceptual domains often approach open number sentences using one of the following ways of reasoning:
  - • Order-based reasoning
  - Analogically based reasoning
  - Computational reasoning
  - Limited ways of reasoning
  - Formal mathematical approach (e.g., logical necessity)
  - Other/unclear



# Students With Negative Integers

- Order-based reasoning
  - Number-line/motion, Grade 7,  $-3 + 6 = \underline{\quad}$

# Students With Negative Integers

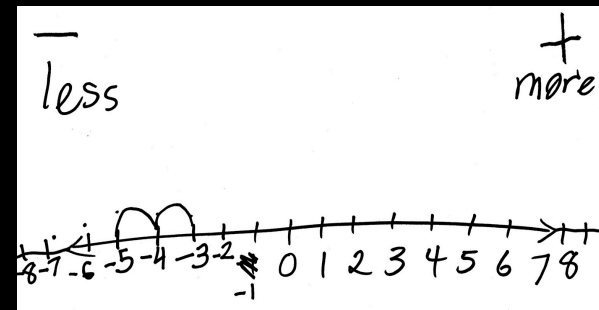
- Order-based reasoning
  - Count by 1s, Liberty, Grade 2,  $-2 + \square = 4$

# Predictions

- Rosie, a second grader, will use a number line and the idea of motion to try to solve each of the following problems:
  - 1)  $\square + 5 = 3$
  - 2)  $-5 - 4 = \square$
  - 3)  $5 + \square = 2$
  - 4)  $3 - \square = -2$
- She correctly solves 3 of the 4. Predict which one she struggled to reason about.

# Rosie, Grade 2, Motion on a Number Line

$$\square + 5 = 3, \quad -5 - 4 = \square, \quad \& \quad 3 - \square = -2$$



Rosie, Grade 2, Motion on a Number Line

$$5 + \square = 2$$

# What's the Difference?

Rosie *could not* solve

$$5 + \square = 2$$

but *could* solve

$$\square + 5 = 3$$

$$\begin{array}{|c|} \hline \text{Starting} \\ \hline \text{Point} \\ \hline \end{array} + \begin{array}{|c|} \hline \text{Change} \\ \hline \end{array} = \begin{array}{|c|} \hline \text{Ending} \\ \hline \text{Point} \\ \hline \end{array}$$

# Next Steps

Think about Rosie who solved  $\square + 5 = 3$ ,  $3 - \square = -2$ , and  $-5 - 4 = \square$  on the number line but could not solve  $5 + \square = 2$  and insisted that  $6 + -2$  is impossible to solve (you did not watch this clip).

What problem(s) might you pose next to extend her thinking?

What meaning is  
assigned to adding or  
subtracting a negative  
number?



$$-3 + 6 \text{ vs. } 6 + -3$$

	$-3 + 6$		$6 + -3$	
	Gr 2 ( $n = 40$ )	Gr 4 ( $n = 40$ )	Gr 2 ( $n = 40$ )	Gr 4 ( $n = 40$ )
3				
Not Possible				
Other incorrect				

$$-3 + 6 \text{ vs. } 6 + -3$$

	$-3 + 6$		$6 + -3$	
	Gr 2 ( $n = 40$ )	Gr 4 ( $n = 40$ )	Gr 2 ( $n = 40$ )	Gr 4 ( $n = 40$ )
3	25%	32.5%		
Not Possible	27.5%	22.5%		
Other incorrect	57.5%	45%		

# $-3 + 6$ vs. $6 + -3$

	$-3 + 6$		$6 + -3$	
	Gr 2 ( $n = 40$ )	Gr 4 ( $n = 40$ )	Gr 2 ( $n = 40$ )	Gr 4 ( $n = 40$ )
3	25%	32.5%	0%	17.5%
Not Possible	27.5%	22.5%	35%	32.5%
Other incorrect	57.5%	45%	65%	50%

$$6 + \square = 4$$

	Gr 2 ( <i>n</i> = 40)	Gr 4 ( <i>n</i> = 40)
-2		
Not Possible or Not Sure		
Other incorrect		

$$6 + \square = 4$$

	Gr 2 ( <i>n</i> = 40)	Gr 4 ( <i>n</i> = 40)
-2	0%	5%
Not Possible or Not Sure		
Other incorrect		

$$6 + \square = 4$$

	Gr 2 ( <i>n</i> = 40)	Gr 4 ( <i>n</i> = 40)
-2	0%	5%
Not Possible or Not Sure	87.5%	92.5%
Other incorrect	12.5%	2.5%

# Students With Negative Integers

- Children who have negative integers in their conceptual domains often approach open number sentences using one of the following ways of reasoning.
  - Order-based reasoning
  - • Analogically based reasoning
  - Computational reasoning
  - Limited ways of reasoning
  - Formal mathematical approaches
  - Other/unclear

# Students With Negative Integers

- Analogically based reasoning
  - Negative numbers thought of as objects, Grade 1,  $-8 -$   
 $-1$



# Students With Negative Integers

- Analogically based reasoning

Negative numbers thought of as \$ owed, Gr 4,  $-3 + 6$

Affordances and  
limitations of an  
analogically based  
approach to negative  
numbers

# James, Grade 1, Treats Negatives Like Positives

$$-7 - \square = -5$$

James, Grade 1, Treats Negatives Like Positives

$$1 + -2 = \square$$

# Next Steps

Think about James who solved  $-5 + -2$  and  $-7 - \square = -5$  but could not solve  $1 + -2 = \square$  or  $-1 + 4 = \square$ .

Why did James struggle with  $1 + -2$  and  $-1 + 4 = \square$ ? What problem(s) might you pose next to extend his thinking?

# Negatives & Positives as Animals in a Zoo

- If negatives are lions and positives are zebras, they must stay in their own cages.
- Of course they don't interact.
- For students like James, the main issue is developing meaning for how to *blend* these numbers.

**-5 + -1 vs. -5 - -3**

	<b>-5 + -1</b>		<b>-5 - -3</b>	
	<b>Gr 2 (n = 40)</b>	<b>Gr 4 (n = 40)</b>	<b>Gr 2 (n = 40)</b>	<b>Gr 4 (n = 40)</b>
<b>Correct</b>				
<b>Not Possible</b>				
<b>Other incorrect</b>				

# $-5 + -1$ vs. $-5 - -3$

	$-5 + -1$		$-5 - -3$	
	Gr 2 ( $n = 40$ )	Gr 4 ( $n = 40$ )	Gr 2 ( $n = 40$ )	Gr 4 ( $n = 40$ )
Correct	37.5%	62.5%		
Not Possible	12.5%	2.5%		
Other incorrect	50%	35%		



# $-5 + -1$ vs. $-5 - -3$

	$-5 + -1$		$-5 - -3$	
	Gr 2 ( $n = 40$ )	Gr 4 ( $n = 40$ )	Gr 2 ( $n = 40$ )	Gr 4 ( $n = 40$ )
Correct	37.5%	62.5%	40%	70%
Not Possible	12.5%	2.5%	12.5%	0%
Other incorrect	50%	35%	47.5%	30%

# Students with negative integers

- Children who have negative integers in their conceptual domains often approach open number sentences using one of the following ways of reasoning.
  - Order-based reasoning
  - Analogically-based reasoning
  - • Computational reasoning
  - Limited ways of reasoning
  - Formal mathematical approaches
  - Other/unclear

# Students with negative integers

- Children who have negative integers in their conceptual domains often approach open number sentences using one of the following ways of reasoning.

- Order-based reasoning
- Analogically-based reasoning
- Computational reasoning
- Limited ways of reasoning
- • Formal mathematical approaches
- Other/unclear

# Students with Negative Integers

- Logical Necessity, Grade 1,  $5 + -2$  cf.  $-2 + 5$

# Students with Negative Integers

Logical Necessity (Part I) Roland, Gr. 4,

$$-5 + -1 = \square$$

# Students with Negative Integers

Logical Necessity (Part 2), Roland, Gr. 4,

$$-5 - -3 \quad \text{cf. } -5 + -3$$

# What we know so far ...

- Young children have ideas about negative numbers that can be leveraged.
- Different ways of reasoning have both limitations and affordances.
- Understanding integers is complex! One way of reasoning is likely insufficient for making sense of integer operations.
- Similar to the historical development of integers, students, too, can use fundamental mathematical principles and formalisms to reason about negative numbers.

# Rankings Revisited

Letter	Item	Ranking	% Correct ( $n = 80$ )
g)	$-5 - -3$	1	55%
d)	$-5 + -1$	2	50%
b)	$3 - 5$	3	28.75%
f)	$-3 + 6$	3	28.75%
e)	$6 + -3$	4	7.5%
a)	$6 + \_ = 4$	5	2.5%
c)	$5 - \_ = 8$	5	2.5%



What have we  
learned about integer  
reasoning while it  
develops over time?

# Big Ideas

## Prior to Formal Instruction

- Many young children have productive ideas about negative numbers.
- Children are sense makers and often engage in *doing* mathematics (Stein et al., 1996).
- Many children are consistent in their approaches to and reasoning about integer problems.
- In young children we see the emergence of reasoning about negative numbers in terms of what we call *logical necessity*.

# Big Ideas

## After Formal Instruction

Most middle school students...

- invoke rules about how to add and subtract negative numbers.
- struggle to explain the rules they invoke.

Teachers' dilemmas include

- how to support student reasoning in concert with developing efficiency
- how to distinguish rule following without understanding from “reasoned” generalizations

# Big Ideas

## Long After Formal Instruction, Successful Mathematics Students

- Most successful high school students invoke a variety of ways of reasoning to solve problems involving integer addition and subtraction.
- Most high school students struggle to reason about the equivalence between subtraction and adding the inverse.
- Many successful high school students have not developed a meaning for “ $-$ ” as negation (or the opposite of).

Successful High School Students have many tools in their integers belt, yet still have room to grow!

Thank You

# Discussion/Questions

# Students With Negative Integers

- Logical Necessity, Grade 2,  $2 - \_ = 6$ ,  $2 - -4$  cf.  $2 - 4$