Seventh-Grade Students' Performance and Ways of Reasoning:

Integer Addition and Subtraction Problems

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Researchers have investigated students' reasoning and performance on integer addition and subtraction problems after school-based instruction (e.g., Bofferding & Richardson, 2013; Chiu, 2001; Murray, 1985). These studies can be grouped into two categories on the basis of the age of the participants: college students (Bofferding & Richardson, 2013; Chiu, 2001) and middle or high school students who had experienced instruction with integers *within two years* (Chiu, 2001; Murray, 1985). In the first category, college students, regardless of major, demonstrated high rates of success on both integer addition and subtraction problems. Bofferding and Richardson (2013) documented that many tended to use rules or order-based reasoning to solve problems, whereas Chiu (2001) found that students used multiple metaphors to explain how they obtained and justified their answers. However, findings on the performance and reasoning of students who had only recently experienced integer instruction contrast with the above findings in two ways. First, Murray (1985) found that students were much less successful solving subtraction problems (46%–69% correct, average 57%) than addition problems (74%–78% correct, average 76%). Second, in interviews with high-performing 9th-grade students, Murray (1985) found that errors were often due to misapplied rules. Thus, researchers found that after recent school-based instruction, participants found integer subtraction problems more challenging than integer addition problems and that errors were often due to misapplied rules.

In our work, we sought to find out to what degree Murray's 1985 findings about students who have recently completed integers instruction have stayed the same and to consider how findings might have changed. Additionally, because of our recent work on problem types (Lamb, Bishop, Philipp, Whitacre, & Schappelle, 2017) and ways of reasoning (WoRs) (Bishop, Lamb, Philipp, Whitacre, & Schappelle, 2017), we sought to apply new frameworks to the problem solutions that this group of students provided. Thus, we sought to answer the following questions:

1) How successful are 7th graders at solving integers addition and subtraction problems, and how do these results compare to Murray's (1985) findings?
2) How often do 7th graders use each of the five ways of reasoning: order-based, analogy-based, formal, computational, and emergent?
3) In what ways, if any, are 7th graders' uses of WoRs related to problem type?
4) What, if any, relationships exist among problem types, WoRs, and performance?

Method

We assessed a cross-section of students from the Southern California region. From across seven school sites in Southern California, 198 students responded to a paper-pencil assessment. The students attended schools that represented a range in socioeconomic status as determined by percentage of students at each site on free or reduced-cost lunch (10%–100%, average 49%). Seven teachers conducted the paper-pencil assessment using detailed protocols for administering the assessment. Students were allowed as much time as they needed to complete the assessment. The paper-pencil survey consisted of the following seven open number sentences and one story problem. We focus our analysis on the open number sentences. See Table 1 for items and performance, sorted by subtraction versus addition.

Table 1
Items and Percentage Correct on Integer Addition and Subtraction Open Number Sentences

Subtraction n = 198	Addition n = 198

Item	Performance	Item	Performance
-5 – -3 = □	65%	-2 + □ = 4	80%
6 – -2 = □	64%	-5 + -1 = □	75%
3 – □ = -9	49%		
5 – □ = 8	41%		
-7 – □ = -4	39%		

Findings

The most challenging problems were the subtraction problems, with an average of 52% of the students answering correctly. In contrast, more than three fourths of the 7th graders correctly answered the two addition problems (average of 78%). These findings are consistent with those from Murray (1985) wherein Murray also found that integer subtraction was more challenging than integer addition (averages of 57% and 76% correct, respectively).

Ways of Reasoning, Problem Types, and Performance

Ways of reasoning. Across the seven open number sentences, more than half of all problems (55%) were solved using a computational way of reasoning (see Table 2). Each of the other ways of reasoning was used in fewer than one fifth of the solutions. In particular, use of the Emergent and Formal WoRs was rare (3% and 2%, respectively).

Table 2
Percentage Use of Ways of Reasoning

Way of reasoning	Percentage use n = 198
Computational	55%
Order-Based	17%
Analogy-Based	14%
Emergent	3%
Formal	2%
Unclear	9%

Problem Types and Ways of Reasoning. Elsewhere, we shared three problem types: Change Positive, All Negatives, and Counterintuitive (Lamb et al., 2017). These problem types are characterized by the signs and locations of the values in the open number sentences. We know that the problem types distinguish performance and degree of use of the ways of reasoning for students prior to their receiving school-based instruction. We were curious about whether we would see similar patterns with 7th-grade students. Thus, one question we sought to explore was whether problem types differentially elicited WoRs for 7th-grade students. If so, we wondered whether the patterns would be similar to patterns of those students who had yet to receive school-based instruction (Lamb et al, 2017). In Table 3, see the problem types and the percentage of problems that were solved using each WoR.

For every problem type, the computational WoR was used more often than any other WoR (an average of 55% of the time). Order-based reasoning was used more often on Change-Positive problems (27%) than on All Negatives (15%) or Counterintuitive problems (10%). Similarly, Analogy-based reasoning was used more often on All Negatives problems (19%) than on Change-Positive (11%) or Counterintuitive problems (9%). Computational reasoning was used almost two-thirds of the time (64%) to solve Counterintuitive problems, more than on Change Positive (46%) and All Negatives (54%) problems. Further, Computational Reasoning was used much more often than any other WoR on Counterintuitive problems. The 2nd most common WoR was Order-based, used only 10% of the time.

Table 3
Percentage Use of Ways of Reasoning for Every Problem Type

Problem Type	Change Positive *n* = 198	All Negatives *n* = 198	Counterintuitive *n* = 198
Open number sentences	-2 + □ = 4 3 – □ = -9	-5 + -1 = □ -5 – -3 = □ -7 – □ = -4	6 – -2 = □ 5 – □ = 8

Way of reasoning	% Use	% Use	% Use
Computational	46%	54%	64%
Order-based	27%	15%	10%
Analogy-based	11%	19%	9%
Formal	4%	2%	1%
Emergent	1%	1%	8%
Unclear	12%	8%	8%

When comparing findings for these 7th-grade students to findings from students who had yet to receive school-based instruction, we found two similarities and two differences. First, for the order-based and analogy-based WoRs, the relationships between problem types and WoRs were consistent in that students used order-based reasoning more often to solve Change Positive problems than the other two problem types, and they used Analogy-based reasoning more often to solve All Negatives problems than the other two problem types. One difference is that for students prior to instruction, the most common WoR for every problem type was the Emergent WoR, whereas for 7th graders, the most common WoR on every problem type was Computational. Further, on Counterintuitive problems, whereas the most common WoR for students prior to instruction was Emergent (76%), the most common WoR for 7th graders was Computational (64%).

Problem types, ways of reasoning, and percentage correct. We were curious to know whether one WoR tended to elicit more accurate performance than the others and whether success, problem type, and WoR were related (See Table 4.) For 7th-grade students, although Formal reasoning was rarely used, its use elicited a high percentage of correct answers on *all* problem types (80–93% correct). In contrast, Emergent reasoning elicited *no* correct answers across all problem types. Computational reasoning elicited correct answers on about two-thirds of the problems (59%-69%) across problem types.

Table 4
*Percentage Correct When Using Each Way of Reasoning for Each Problem Type**

Way of reasoning	Change Positive	All Negatives	Counterintuitive
	% Correct	% Correct	% Correct
Overall	64%	60%	53%
Formal	**93%**	**91%**	**80%**
Analogy-based	61%	**85%**	36%
Order-based	**75%**	46%	28%
Computational	61%	59%	**69%**
Emergent	0%	0%	0%
Unclear	52%	34%	16%

*These percentages reflect the percentages of problems (within each problem type) answered correctly from among the number of problems (within each problem type) solved using that WoR. For example, in Table 3, one can see that 4% (or 14 of 396) of the Change Positive problems were solved using Formal reasoning. Of those 14 uses, 93% (or 13) resulted in correct responses.

Order-based reasoning and analogy-based reasoning differentially elicited accurate performance across problem types. Whereas three fourths of the Change-Positive problems solved using Order-based reasoning were answered correctly, only about one half and one fourth of the All Negatives and Counterintuitive problems were answered correctly using order-based reasoning. Similarly, 85% of All Negatives problems were answered correctly when using Analogy-based reasoning, but about three fifths and one third of the Change-Positive and Counterintuitive problems were answered correctly when students used Analogy-based reasoning.

These findings show that problem types and WoRs interact. Order-based reasoning was used more often on Change-Positive problems than on problems of other types, Analogy-based reasoning more often on All Negatives problems than on problems of other types, and

Computational reasoning more often on Counterintuitive problems than on problems of other types. Additionally, success rate improved when these particular WoRs were used for the particular problem types, and success rates were higher for students using that WoR than for students using all other WoRs except for Formal.

Summary

We share four findings. First, 7th-grade students were much less successful with integer subtraction problems than integer addition problems, consistent with Murray's (1985) findings from 30 years ago. Second, more than half of all problems were solved using a computational approach, and the three most common ways of reasoning were Computational, Order-based, and Analogy-based. Use of Formal and Emergent WoRs was rare. Third, ways of reasoning were used differentially and in predicted ways on problem types. Order-based reasoning was used more often on Change Positive problems than on problems of the other two types, Analogy-based reasoning was used more often on All Negatives problems than on problems of the other two types, and Computational reasoning was used more often on Counterintuitive problems than on problems of the other two types. Fourth, when WoRs were used to solve problems in the predicted ways, the use of that WoR elicited higher percentages of correct answers than when that WoR was used to solve problems of the other two types (75% of Change Positive problems were solved correctly when students used Order-based reasoning, 85% of All Negatives problems were solved correctly when students used Analogy-based reasoning, and 69% of Counterintuitive problems were solved correctly when students used Computational reasoning). The use of Formal reasoning always led to high percentages of correct responses.

The findings confirm that the use of Computational reasoning is common among 7th-grade students and that integer subtraction is more challenging than integer addition for students in this grade. However, we found consistent patterns between the use of Order-based reasoning and Analogy-based reasoning on Change Positive and All Negatives problems, respectively. Further, we showed that students perform better when they use those WoRs to solve the particular problem types.

Limitations

We collected these data prior to the identification of problem types, and so if we were to conduct this study again, we would use problems such as those in Table 5 to account for addition and subtraction for *each* problem type, along with both result- and change-unknown problems. Also, in this assessment we did not include problems with sums or differences of 0, and we would include those, too. Further, we recognize that assessing students using a paper-pencil assessment has limitations. For example, because we did not interview students, we could not pose follow-up questions when strategies shared were unclear. Thus, 9% of the strategies were identified as Unclear. That said, we were able to gather data from a relatively large sample across several sites, and so one would have to weigh benefits, drawbacks, and goals when determining the most appropriate method of assessing students' mathematical ideas.

Table 5
Proposed Problem Set for Future Assessments

Problem type	Change Positive		All Negatives	Counterintuitive
	Cross 0	Do Not Cross 0		
Addition				
Result unknown	-3 + 6 = □	-8 + 6 = □	-5 + -1 = □	6 + -3 = □
Change unknown	-2 + □ = 4	-9 + □ = -4	-5 + □ = -8	6 + □ = 4
Subtraction				
Result unknown	4 – 7 = □	-2 – 7 = □	-5 – -3 = □	6 – -2 = □
Change unknown	3 – □ = -9	-2 – □ = -8	-7 – □ = -4	5 – □ = 8

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