High School Students' Conceptions of the Minus Sign[1]

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Abstract

The minus sign elicits three useful meanings for students: subtraction (5 – 3), part of the symbol for a negative number (-4), and the *opposite of* (--4, the opposite of negative 4). High School students reason productively about the first 2 meanings but rarely appear to invoke the 3rd meaning.

In what ways would your students reason about the items in Table 1? Would they apply the rules that are commonly shared in mathematics textbooks? Would they use equations to solve Items 1–5? How might they approach Items 6–8?

Table 1. *Problems Posed to High School Students*

Item
1) $5 + \square = 4$
2) $5 - \square = 8$
3) $\square + 5 = -2$
4) $\square - 5 = -1$
5) $\square - 5 = -8$
6) Compare   --4   and   -4
7) Compare   -x   and   x
8) *In one textbook we found this definition of absolute value. Can you explain this to me?* $$\lvert x \rvert = \begin{cases} x \ \text{if } x \geq 0 \\ -x \ \ \text{if} \ \ x < 0 \end{cases}$$

We gave these tasks and others to high school students and found that students productively and creatively solved the first five tasks. However, we also found that Tasks 6–8 were more challenging for students than we expected. We believe that students' struggles on these problems may be attributed to their limited conceptions of the minus sign. When reflecting on the students' responses, we realized that, as teachers, we rarely helped our students make explicit different meanings of the minus sign, but we believe that focused attention on these meanings could support students' productive reasoning on *all* eight tasks. In the next section we explain three meanings of the minus sign, share highlights of the students' responses to the eight tasks, and suggest implications for instruction.

In Table 2, all three problems make use of the symbol "–," to which we refer, in general, as the minus sign. Yet each problem may elicit a different meaning for students. In Problem 1, the minus sign indicates subtraction (a binary operation, or a function with exactly two operands), whereas in Problem 2, the minus sign is part of the symbolic representation for a negative number. In Problem 3, however, the first minus sign may be thought of as the *opposite of* (a unary operator, or a function with exactly one operand) so that one could read - -4 as "the opposite of negative four" rather than students' more common reading of "negative negative four" (see Bofferding, 2010; Vlassis, 2008). This *opposite of* interpretation is a powerful way of conceiving the minus sign, but we have found that few students, including high school students, appear to hold this conception.

Table 2. *Three Meanings of the Minus Sign*

Problem	A Possible Meaning of the Minus Sign
1. 3 – 5 = ☐	Subtraction (a binary operation)
2. -1	Symbolic representation for a negative number
3. Which is larger, - -4 or -4?	The opposite of (a unary operation)

The Importance of the Three Meanings of the Minus Sign

Students may initially face difficulties making sense of the three meanings. However, having one symbol represent several ideas streamlines calculations and procedures in many ways. First, we often treat one meaning of the symbol as if it were another when calculating answers or simplifying expressions. For example, some mathematically proficient students will claim that they *see* the problem -4 – 7 as having two negative numbers (-4 and -7) and will appropriately but implicitly reason about the problem as -4 + -7. Students who reason this way rarely express that the problem as originally stated includes one negative number and a subtraction of a positive number, even when prompted for an explanation. However, this ability to treat one meaning of the sign as if it were another is efficient for making calculations.

Second, the meaning of the minus sign changes during the process of solving equations (Vlassis, 2008), and students need to be able to move between the two interpretations. For example, in the problem $3 - x = 8$, the initial meaning of the minus sign is subtraction. When one solves for x by first subtracting 3 from both sides and writes $-x = 5$, that action necessitates a change in the meaning of the sign from subtraction, to the *opposite of*. Students who recognize the minus sign as sometimes meaning the *opposite of* could reason that because the opposite of x is 5, then x must be equal to -5.

Third, the notion of the minus sign as the *opposite of* may support deep understanding of variables, algebraic expressions, and symbolically represented definitions. For example, this meaning can support students' understanding of the definition of absolute value, the definition of odd functions, $f(-x) = -f(x)$, or help students to make sense of algebraic expressions such as $-x$. Students who conceive of the expression $-x$ as the *opposite of x* may be more likely than students who hold only a *negative x* conception to understand that $-x$ may represent positive numbers.

Supporting students' sense making of the three meanings of the minus sign and when each might be invoked is important but often overlooked. For some problems, students can use multiple meanings of the minus sign and still reason appropriately about the problem. For other problems, such as #3 in Table 2, students must invoke the notion of the *opposite of* to reason robustly about the problem. To develop mathematically sophisticated ideas, students need to learn the different meanings and understand when the meanings shift during problem solving.

Background

In the next sections, we share students' responses to tasks that make use of the various meanings of the minus sign. In sharing these responses, we hope to illuminate students' varied and effective ways of reasoning about the minus sign as both the sign of a negative number and as subtraction; we also share responses that show that students may have fragile notions of the *opposite of* meaning of the minus sign.

Students

We interviewed nine high school students from a large suburban high school. All students had passed geometry and six of the nine had taken either an honors mathematics course or an advanced placement mathematics course; we thus deemed these students to be among the more successful high school mathematics students at the school. Each interview lasted approximately one hour and consisted of problems designed to provide evidence of students' reasoning about negative numbers and the meanings of the minus sign.

Interview Items

The interview comprised a variety of problems, but the bulk of the interview consisted of a set of open number sentences and comparison problems (see Table 1 for sample questions). Problems 1–5 in Table 1 are identified as open number sentences for which the unknown is

located on the left side of the equation. The comparison problems consisted of pairs of numbers or algebraic expressions. For each pair, students were asked to circle the larger, write an equal sign between the two if they were equal, or indicate with a question mark that there was insufficient information to determine the larger value. Problem 8 required students to interpret the meaning of a formal, symbolic definition of absolute value that may elicit students' use of the *opposite of* meaning of the minus sign. In addition to providing written responses, students were asked to explain their reasoning.

Results

In general, students were successful in solving the open number sentences. Collectively, the students in our study correctly solved 91% of the open number sentences in Table 1. However, the strategies students used were often *not* the rules and procedures emphasized in textbooks. For example, of the 45 explanations the nine students shared of their thinking about the five open number sentences in Table 1, students invoked a typical rule such as changing subtraction of a negative number to addition of a positive number only 7 times. We were surprised to find that students instead invoked the first two meanings of the minus sign, subtraction and the sign of the negative number, and often moved flexibly between these meanings. However, students almost never invoked the *opposite of* meaning. In the next sections we briefly describe students' approaches and highlight the more prevalent and surprising strategies students utilized.

Students' Reasoning About the Minus Sign as the Sign of a Negative Number

On Problem 3, $\Box + 5 = -2$, the nine students used seven different strategies, and most invoked the meaning of the minus sign as the sign of a negative number. For example, one student compared the magnitudes (absolute values) of the numbers while simultaneously

attending to the sign of the number. This student reasoned that because -2 is indeed negative and that the 5 is positive, the unknown had to be a "number larger than 5 but in negative form." Other students used a number line, considering where to start and how to move on the basis of the given operation and signs of the numbers. On this problem, the students generally invoked the meaning of the minus sign as referring to the *sign of a negative number*, sometimes thinking of the negative number as a number less than zero and sometimes as a location on a number line.

On Problem 1, $5 + \square = 4$, all nine students examined the relative sizes of the numbers to help them solve this problem. In general, they reasoned that the missing addend had to be a negative number because when one adds 0 or a positive number to 5, the sum should be equal to or larger than 5, whereas in this problem the sum is only 4. We were surprised to find that across all five open number sentences, students invoked reasoning that involved determining the number's sign prior to determining the magnitude in almost 40% of the explanations. Additionally, five of the nine students referred to this problem as being like *subtracting* 1 from 5, demonstrating flexibility in treating the minus sign as subtraction, even though the problem as written involves addition.

Students' Reasoning About the Minus Sign as Subtraction

Three of the problems in Table 1 explicitly include the minus sign as subtraction and, unsurprisingly, students were articulate about expressing this meaning of the minus sign as subtraction. For example, on the problem $\square - 5 = -1$, one student reasoned that $5 - 5 = 0$ and that when the unknown is larger than 5, then the difference is a positive number. Thus, because the difference was -1, the unknown had to be less than 5, and, in this case, one less than 5, or 4. On Problem 2, $5 - \square = 8$, several students correctly reasoned about the minus sign as subtraction by noting that because the problem was subtraction but 8, (the difference), was larger than 5, (the

minuend), that the unknown *had* to be negative, and thus the answer was -3. Their explanations demonstrated that they invoked the minus sign as subtraction, and, when they determined that the answer was negative, also as the sign of the number.

Students' Reasoning About the Minus Sign as the Opposite of

The students were less successful on problems that required them to (at least implicitly) conceive of the minus sign as the *opposite of* (a unary operator). For example, although six of the students correctly compared - -4 and 4, five of the six invoked a rule that two negatives make a positive and were unsure why the rule was correct. Only *one* of the nine students explicitly described - -4 as meaning the opposite of -4, or 4. Additionally, five of the students appropriately compared –x and x, but four of the five again invoked the rule that two negatives make a positive (these students tested cases of positive and negative numbers). The four who incorrectly answered assumed that *x* is always larger than –x because –x always represents a negative number. Although some students evaluated these problems correctly without an explicit *opposite of* interpretation (by either noting that two negatives make a positive or that -1 times a negative number is a positive number), we believe that students need to be able to reason more robustly than demonstrated with that kind of response. If students' *only* way of justifying that - - 4 is larger than -4 is because two negatives make a positive, then this potentially more narrow understanding of the minus sign could create gaps in their sense making that surface when transitioning to more advanced mathematics. We see evidence for these gaps when students are asked to interpret the definition of absolute value.

Only two students who were asked to explain the textbook definition of absolute value $|x| = \begin{cases} x \ if x \geq 0 \\ -x \ if \ x < 0 \end{cases}$ were able to clearly demonstrate that the definition made sense; one of the two was the student who had an interpretation of the minus sign as the *opposite of*. One student,

Alan, captured the sentiments of the remaining students when he stated, "No, [this definition] doesn't make sense because even if x is a negative number, then the outcome is still a positive number, so the absolute value of x couldn't be negative x." Presumably Alan thought that $-x$ represents a negative number even when x itself is negative. Another student confidently stated that she understood the definition but when asked to use the definition to determine the absolute value of negative 2, became confused, saying, "I know that [the absolute value of negative 2] is 2, but according to this (points to the definition), it would be negative 2, so I guess it [the absolute value of negative 2] is negative 2."

Summary and Implications for Practice

Every student used strategies that are rarely emphasized in textbooks, and these alternative strategies were almost always invoked appropriately. Students appeared to select a strategy on the basis of the operation and relationship between the numbers; their strategies required them to invoke the meaning of the minus sign as either subtraction or the sign of a negative number. Students also routinely but implicitly moved between the meanings of the minus sign as subtraction and the sign of the number. This movement enabled students to find efficient and varied ways to solve problems. We were surprised by the flexibility in strategies that students demonstrated given that most typical textbook adopt a single-strategy, rule-based approach for operating with negative numbers.

However, a different story emerged in students' use of the minus sign as the *opposite of,* inasmuch as only *one* of the nine students we interviewed ever explicitly invoked the meaning of the minus sign as the *opposite of*. We suspect that this limited view of the minus sign, observed in most students we interviewed, hinders their ability to understand expressions related to symbolically represented definitions (such as the definition of absolute value) or appropriately

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make sense of algebraic expressions such as -*x*. Below we offer three ideas for supporting students in coming to better understand these three meanings of the minus sign.

Pose Open Number Sentences to Encourage Sense-Making in Solving Problems

Teachers can support students' use of the meanings of the minus sign (promoting their adoption of a flexible orientation toward solving problems) by posing various open number sentences. Open number sentences in which the unknown is located on the left side of the equation (as in $\square$ – 5 = 8), may encourage students to reason rather than solely rely on a set of memorized rules. We also suggest that teachers use a blank, $\square$, instead of a more traditional variable, *x*, when posing these problems, because we have found that when students see equations with variables, they tend to perform a series of rule-based steps rather than reason about the relationships among the numbers in the problem. Furthermore, teachers might challenge students to solve these number sentences without applying the typical algebraic equation-solving approach. As teachers ourselves, we have observed that students have the capability and the desire to reason through these types of problems rather than to search for that *one way* to get an answer.

Use Tasks to Elicit the Meaning of the Minus Sign as the Opposite Of

Students rarely explicitly conceive of the meaning of the minus sign as the *opposite of* but may do so when using a set of tasks such as a-f below. Tasks a, d, and e are compare problems. Students circle the larger of the two quantities, or put an equal sign if the two quantities are equal. If there is not enough information to determine the larger, students should put a question mark between the two.

a)   - -8   8

b)   Explain what - - - 4 means.

c)   Explain what - - - - 7 means.

d)   *x*   -6

e)   *x* $-x$

f)   *In one textbook we found this definition of absolute value. Can you explain this to me? How would you use this definition to explain how to find the absolute value of -8?*

$$|x| = \begin{cases} x \ if \ x \geq 0 \\ -x \ if \ x < 0 \end{cases}$$

g) $12 = -\underline{}$ *If possible, fill in the blank to make a true statement.*

By considering Problems a)–c), students can practice reasoning about the minus sign as the *opposite of*. Further, in analyzing Problem d), students can expand their conceptions so that the possible values for *x* include negative numbers. Students can discuss how, depending on the value of *x*, either expression may be larger. After students have reasoned through several problems similar to a)–d), teachers can pose tasks like e)–g) to provide students with opportunities to expand their ideas about the minus sign by combining ideas about possible values for *x* with the meaning of the minus sign as the *opposite of*.

Discuss Strategies to Highlight the Meanings of the Minus Sign

Our last instructional implication involves discussing the variety of strategies that students use in order to engage students in explicit conversations about the meanings of the minus sign. Although students we interviewed did alternate between the first two meanings of the minus sign, they often made these shifts implicitly (for example, noting that -4 – 7 has two negatives). Further, using the language of *opposite of* may support students' understanding of different types of functions. For example, asking students in advanced mathematics classes to characterize odd and even functions ($f(-x) = -f(x)$ and $f(x) = f(-x)$, respectively) using the various meanings of the minus sign may help them to clarify their own thinking about these

functions. Engaging in conversation about strategies and being pressed for explanations will help students make explicit the use of the minus sign they are invoking.

We believe that students can build on their knowledge of the meanings of the minus sign to develop a more complete understanding of these three meanings. Discussing explicitly each meaning of the minus sign will provide students opportunities to develop more sophisticated understandings about operations, algebraic expressions, and symbolic definitions. Teachers who become aware of students' conceptions will likely make room for rich class discussions whenever these opportunities arise.

References

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