Developing Symbol Sense for the Minus Sign

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Abstract

Research on how students make sense of and use the minus sign indicates that students struggle to understand the multiple meanings of the minus sign. Teachers can support students in developing robust understanding of each interpretation of the minus sign, and associated symbol sense, so that they can move among the meanings and can recognize and use each meaning seamlessly.

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Imagine that you are teaching a class and ask your students to read this equation:

$$-x - 9 = 12.$$

What would your students say? If you replied, "Negative x minus 9 equals 12," then that

is consistent with what most students would say. And yet students could read this

equation in a way that could convey a different, even deeper, understanding. In this

article we highlight three meanings of the minus sign and explore specific difficulties

related to the minus sign that many students experience. We also suggest ways that you

as a teacher might support your students in developing robust understanding of the minus

sign.

What research has revealed about how students interpret and make use of the

minus sign is a facet of the work on *symbol sense*, which Arcavi (1994) described as

"making friends with symbols" (p. 25), including an understanding and feel for symbols,

how to use them and read them. Research indicates that many middle school, and even

high school, students do not have fully developed symbol sense in relation to the minus

sign. In our research we have learned that this limited view of the minus sign interferes

with students' abilities to truly understand the process of solving equations and make

sense of variables.

Three Meanings of the Minus Sign

There are three common usages of the minus sign. All three problems in Table 1

make use of the symbol "−" that we refer to, in general, as the minus sign. Yet each

problem may elicit a different meaning for students. In Problem 1, the minus sign

indicates subtraction, the first use of the symbol that children encounter. Whereas in

Problem 2, the minus sign is part of the symbolic representation for a negative number—

in this case, "negative two." In Problem 3, however, the first minus sign may be thought

of as the *opposite of* so that one could read - -4 as "the opposite of negative four" rather

than students' more common reading of "negative negative four" (see Bofferding, 2010;

Vlassis, 2008).

Table 1. *Three Meanings of the Minus Sign*

Problem	Meaning of the minus sign
1. $5 - 8 = \square$	Subtraction (a binary operation)
2. $\square + 5 = -2$	Symbolic representation for a negative number
3. Which is larger, **-** -4 or -4?	The opposite of (a unary operation)

In Problem 1 the minus sign is used as a binary operator. A *binary operator* is a

function that takes two inputs and results in one output. Subtraction is a binary operator

because the inputs (5 and 8) result in one output (-3). Addition, subtraction,

multiplication, and division are examples of binary operators because they may be

thought of as functions that involve two inputs and one output. In contrast, the minus sign

is used as a unary operator in Problem 3. A *unary operator* involves only one input and

one output. When one thinks of **-** - 4 as "the opposite of negative 4," then one is viewing

the first minus sign as the unary operator, the *opposite of.* Note also that some people

may think of the minus sign in -2 (see Problem 2) as a unary operator, where one does

not view the minus sign as part of the number but instead as the *opposite of* 2. One who

can view -2 in both ways can be said to flexibly hold both meanings of the minus sign.

Although students may initially face difficulties because three meanings are

assigned to the same symbol, having the same symbol represent several ideas is

important. First, we often treat one meaning of the symbol *as if* it were another. For

example, mathematically proficient students and adults will often rewrite $3 - (x + 5)$ as 3

$- x - 5$ without pause. They implicitly invoke two mathematical principles: (a) the equivalence of subtraction and addition of the subtrahend's inverse, and (b) the distributive property of multiplication over addition (see Figure 1). Students and adults who reason this way rarely explicitly identify that they have treated the subtraction symbol as if it were indeed a negative sign. However, this ability to treat one meaning of the sign as if it were another is efficient for making calculations.

$3 - (x + 5) = 3 - x - 5$ *because*	
$3 - 1(x + 5) = 3 + -1(x + 5)$	subtracting a is equivalent to adding $-a$;
$3 + -1(x + 5) = 3 + -x + -5$	distributive property of multiplication over addition;
$3 + -x + -5 = 3 - x - 5$	adding $-a$ is equivalent to subtracting a.

Figure 1. Justification for the equivalence, $3 - (x + 5) = 3 - x - 5$.

Second, the meaning of the minus sign can change during the process of solving equations (Vlassis, 2008). For example, in the problem $5 - x = 12$, the initial meaning of the minus sign is subtraction. When one solves for x by subtracting 5 from both sides, writing $-x = 7$, that action necessitates a change in the meaning of the minus sign from a binary operator to a unary operator (the *opposite of*). Students who recognize the minus sign as sometimes meaning the *opposite of* could reason that because the opposite of x is 7, then x must be equal to -7. For students to truly *understand* the process of solving equations, they need to be able to flexibly move among the different interpretations of the minus sign and be aware of when and why they can change interpretations.

Third, the notion of the minus sign as the unary operator *opposite of* is critical in supporting sophisticated understanding of variables, algebraic expressions, and symbolically represented definitions. However, research indicates that middle and high

school students have difficulty on problems that require them to (at least implicitly) conceive of the minus sign as the *opposite of*. For example, in our research, we found that only about one-fourth of middle school students interviewed correctly identified - -4 as larger than -4. Further, they did not recognize that they had insufficient information to determine the larger of -*x* or *x*. Conceiving of the minus sign as representing a negative number or as representing subtraction is not particularly helpful for these types of problems.

We take the students' responses to these problems as evidence that they not only struggle to respond correctly but also, by and large, have not engaged with the meaning of the minus sign as the *opposite of*. Students who conceive of the expression -*x* as the *opposite of x* may be poised to understand that -*x* may represent a positive number. But students who lack a strong conception of -*x* and *opposite of* can, understandably, struggle. When students use the language "negative *x*" to read or say "-*x*," they may think that "-*x*" should represent a negative number. Negative is used in the term, so why would one expect that "negative *x*" could represent a positive number?

Although we have found that students can move between the meanings of the minus sign as subtraction and as the sign of the number, students often *implicitly* make these shifts and struggle to explain their thinking. Further, very few students invoke the meaning of the minus sign as the *opposite of*. This limited view of the minus sign hinders students' abilities to understand expressions related to symbolically represented definitions (such as the definition of absolute value) and appropriately make sense of algebraic expressions (such as -*x*).

Supporting Minus-Sign Symbol Sense in the Classroom

We have come to believe that supporting students' sense making of the three meanings of the minus sign and when each might be appropriately invoked is important but nontrivial. Learning the different meanings of the minus sign, recognizing the appropriate meaning in a problem, understanding when the meaning shifts during problem solving, and flexibly moving among the meanings requires time and explicit attention. In the following sections we share three ideas for how you can support students to better understand the three meanings of the minus sign and to help students become aware of the shifting meanings.

Make Explicit the Many Meanings of the Minus Sign Through Discussion

Figure 2 provides two examples of problems that may arise naturally in your classroom and how you might leverage those problems to explicitly support students' awareness and understanding of the multiple meanings of the minus sign. In the first example, a student shares that -3 – 5 has two negative numbers, even though this problem as stated contains one negative number and subtraction of a positive number. Pressing students to explain their strategies will draw out both interpretations for discussion. Similarly, when *-x* appears in a problem or as part of a solution (see second example in Figure 2), you could stop and ask the students to verbally read the expression. Many will likely refer to it as "*negative x.*" You might then ask students how they would read the expression "-5." We expect that students will respond, "*Negative 5.*" If students think of and read *-x* as "negative *x*," we suspect that when asked to place *-x* and -5 on the number line, they will place *-x* on the same side of the number line as they place -5. Using the

line of questioning suggested in the second part of Figure 2, the *opposite of* interpretation

can be discussed in conjunction with the *negative-number* interpretation.

Student Work	Teacher Questions
A student remarks, "-3 – 5 has two negatives so my answer is -8."	*I see one negative (point to -3) and a subtraction sign (point to the subtraction sign). Can you explain how you see two negatives?*
Teacher sees that a student has written "-5" and "-x" on the board and asks the following.	*1) What would you call this symbol [point to -5]? Can you place this on the number line?* *2) What would you call this symbol [point to -x]? Can you place this on the number line?* *3) Is there another place we could locate -5? Is there another place we could locate -x? In what ways do these two symbols [point to the minus sign in each expression] have the same meaning? In what ways do they have different meanings?* *4) Could we place the opposite of x [-x] on the positive side of the number line? Why or why not?*

Figure 2. Questions that highlight minus-sign meanings.

Use Opposite Of, Subtraction, *and* Negative *Language With Students.*

     For students to truly *understand* the process of solving equations, they need to

flexibly move among the different interpretations of the minus sign and be aware of when

and why they can change interpretations. Being aware of the language you use can

support students' understanding and help them recognize that the meaning of the minus

sign can change during the equation-solving process. Figure 3 provides suggestions for

explicitly helping students become aware of language use in relation to the minus sign.

For example, in the problem $6 - x = 9$, the initial meaning of the minus sign is

subtraction. If one subtracts 6 from both sides (as part of solving the problem for x), the

resulting equation is $-x = 3$, necessitating a change in the meaning of the sign from

subtraction to the *opposite of*. In the final answer ($x = -3$), the minus sign represents the

sign of a number.

Equation	Teacher Moves
6 Ⓞ x = 9	Point to the minus sign and remind students that the minus sign means subtraction.
Ⓞ x = 3	Point to the minus sign and state the new equation with the language, the *opposite of x* is equal to 3.
x = Ⓞ 3	Point to the minus sign and remind the students that the minus sign can also mean the sign of the number.

Figure 3. Using minus-sign language with students.

Comparison Tasks That Support the Opposite Of *Interpretation of the Minus Sign*

The tasks in Figure 4 can be used to support students in making meaning of the minus sign as the *opposite of*. In discussing Problems a) and b), students can reason about the minus sign as the *opposite of* by comparing numerical expressions. By using Problems c) and d), students can expand their conceptions so that the possible values for x include negative numbers. Students can discuss how, depending on the value of x, either expression may be larger. After students have reasoned through several problems similar to a–d, you can pose a task such as e) to provide students opportunities to expand their ideas about the minus sign by combining ideas about possible values for x with the meaning of the minus sign as the *opposite of*.

Tasks		Directions for Each Task
a) - -4	4	For each, have students circle the larger or write an equal sign if the values are equal. Have students write a question mark if there is not enough information to determine the larger. After each task, discuss how students determined the larger and which meaning of the minus sign they used and why.
b) - 6	- -6	
c) -4	x	
d) $x + x$	x	
e) x	$-x$	
f) 5 = _-_		Ask, "Is there anything that you can put in the blank to make this equation true?"

Figure 4. Tasks to support the *opposite of* meaning of the minus sign.

In closing, we believe that students, given opportunities to engage in discussions like those suggested in Figures 2 through 4, can develop sophisticated understanding of

the three meanings of the minus sign. Explicitly discussing each meaning of the minus sign with a particular emphasis on the *opposite of* conception will enable students to robustly reason about operations and algebraic expressions. We believe that these experiences will support students in developing symbol sense in relation to the minus sign that will foster their future learning as they move from middle school into high school and beyond.

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