An Analysis of Mathematical Content Knowledge for Teaching

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Two Tasks

Below is the work of Terry, a second grader, who solved this addition problem and this subtraction problem in May.

Problem A

$$\begin{array}{r} {\scriptstyle 1} \\ 259 \\ +\ 38 \\ \hline 297 \end{array}$$

Problem B

$$\begin{array}{r} 3 \\ \not{4}{\scriptstyle 1}29 \\ -\ 34 \\ \hline 395 \end{array}$$

- Does the 1 in each of these problems represent the same amount? Please explain your answer.
- Explain why in addition (as in Problem A) the 1 is added to the 5, but in subtraction (as in problem B) 10 is added to the 2.

In March, Andrew, a second grader, solved 63 - 25 = ☐ as shown below.

$$\begin{array}{r} 63 \\ -25 \\ \hline -2 \\ 40 \\ \hline 38 \end{array}$$

- Explain why Andrew's strategy makes mathematical sense.
- Please solve 432 - 162 = ☐ by applying Andrew's reasoning.

Activity

- Take a few minutes to think about the two tasks.
- Think about how you solved each task. What knowledge did you draw upon in solving them?
- If time remains, talk to a neighbor.

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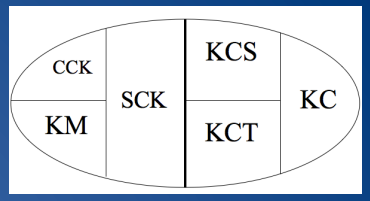
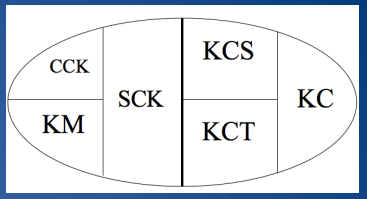
Types of Knowledge



Hill, Ball, Schilling

- Common content knowledge (CCK)
  - The knowledge commonly taught in school and used in life.
- Specialized content knowledge (SCK)
  - The knowledge that teachers need, not because they teach it to students but because it serves as the foundation for common content knowledge.
- Pedagogical content knowledge (PCK)
  - The knowledge at the intersection of the content and students' thinking.

Types of Knowledge: Examples



Hill, Ball, Schilling

- Common content knowledge (CCK)
  - *Evaluate and understand the meaning of 12 ÷ 3.*
- Specialized content knowledge (SCK)
  - *Write a real-life story problem that could be represented by the expression 12 ÷ 3.*
- Pedagogical content knowledge (PCK)
  - *How might children think about the problem you wrote?*

STEP

STUDYING TEACHERS' EVOLVING PERSPECTIVES

Principal Investigators	Randy Philipp, PI Vicki Jacobs, co-PI
Faculty Associates	Lisa Clement Lamb Jessica Pierson Bishop
Graduate Student	John (Zig) Siegfried
Research Associate	Bonnie Schappelle
Project Coordinator	Candace Cabral

Funded by the National Science Foundation, ESI-0455785

STEP Participant Groups

(30+ per group)

K–3 Teachers

Prospective Teachers (PST)	Undergraduates enrolled in a first mathematics-for-teachers content course
Initial Participants (IP)	0 years of sustained professional development
Advancing Participants (AP)	2 years of sustained professional development
Emerging Teacher Leaders (ETL)	At least 4 years of sustained professional development and some minimal leadership activities
Strong Mathematics Students (SMS)	STEM students, with no teaching intentions, enrolled in upper–division mathematics courses

Mathematical Understanding in the Andrew Task

Score	Evidence shown
0	No evidence of mathematical understanding
1	Primarily procedural knowledge; none-to-minimal evidence of conceptual understanding
2	Procedural knowledge and conceptual understanding evident
3	Strong procedural knowledge and conceptual understanding evident; justification incomplete
4	Strong and related procedural knowledge and conceptual understanding evident; justification complete

Andrew and Ones Tasks

Which score for which group?

Group	Andrew	Ones
PST	?	?
IP	?	?
~~AP~~	X	X
ETL	?	?
SMS	?	?

Each response was scored on a 5-point scale (0–4).

Andrew	Ones
1.48	0.31
1.69	0.97
2.53	1.58
2.56	2.72

Andrew and Ones Tasks

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PST	1.69	?
IP	1.48	?
~~**AP**~~	X	X
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Each response was scored on a 5-point scale (0–4).

? Surprised? ?

Findings

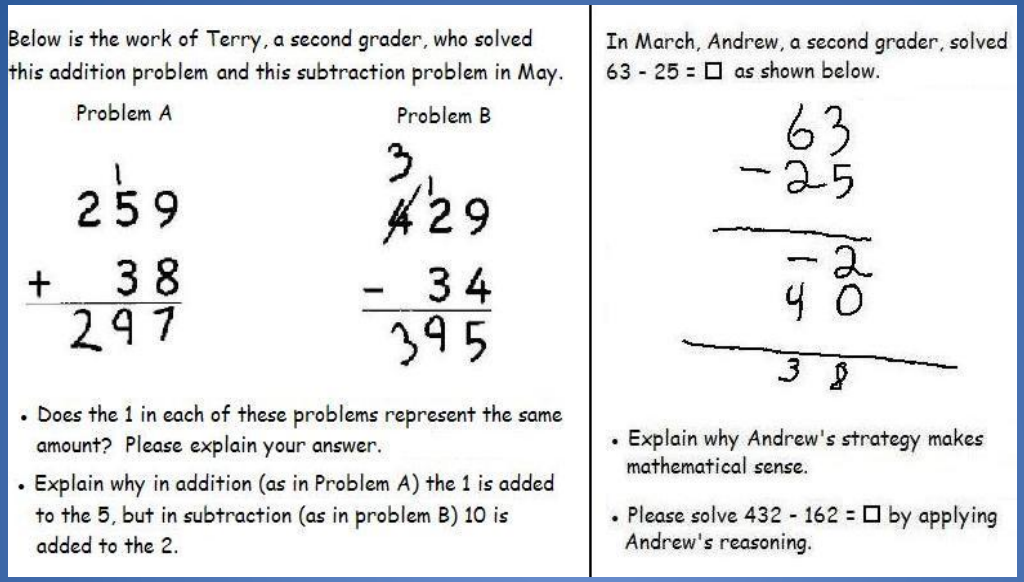
- We posed eight tasks to the participants.
- On six tasks, the SMSs performed as well as the ETLs.

A Second Problem on Which SMSs Struggled

Please provide solution strategies—as many as you can—that you might expect children to use to solve the following problem:

Pablo read 15 pages of his library book on Saturday. The book has 32 pages. How many pages will he have to read on Sunday to finish his book?

Ones Task



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Whereas SMSs cannot answer most tasks, such as the Andrew Task, without grappling with the meaning of the dash 2, they could answer the Ones Task in a purely calculational way without grappling with the place-value ideas embedded in the task. So, for example, they see the 1 in Problem A as a 1 added to the 5 and 3, and they see the 1 in Problem B as a 10 that creates 12 from which 3 is subtracted. By the way, this task presents implications for those of us teaching college students: To have students grapple with underlying concepts, at least initially, constrain the tasks so that they cannot be answered without unpacking the meanings.

Findings

- The SMSs scored as well as the ETLs on the other six tasks.
- Does that show that they hold the same Mathematical Content Knowledge?

The Land of SCK

How do people reach the land of specialized content knowledge?
Does the path one takes to get to SCK have implications for teaching?

The Land of SCK

A Mathematical Road to SCK

A Road to SCK Through Children's Mathematical Thinking

Project Z: Mapping Developmental Trajectories of Students' Conceptions of Integers, 2009–present

Principal Investigators	Lisa Lamb, Jessica Bishop, & Randolph Philipp
Faculty Associate	Ian Whitaker
Graduate Students	Spencer Bagley & Casey Hawthorne
Project researchers	Bonnie Schappelle, Mindy Lewis, & Candace Cabral
Undergraduate Students	Kelly Humphrey, Jenn Cumiskey, Danielle Kessler

Funded by the National Science Foundation, DRL-0918780

San Diego State University

Ways of Reasoning (WoR)

Solve the following and remember how you thought:

- -2 + □ = 4
- -5 + -1 = □
- $5 - \square = 8$

Teachers' Approaches to Integer Tasks

Teachers possess these rich ways of reasoning (WoR) about integers.

They are

- Able to flexibly apply WoR and
- Able to analyze problems and strategically combine WoR.

Kalani: $-3 - \square = 2$

*First, Kalani thinks of moving from -3 to 2, as if the number sentence were -3 + **x** = 2.*

"So I am thinking about the number line …. So I am starting somewhere … and what do I do to end up at positive 2? I am moving 1, 2, 3, 4, 5—five units to the right."

Second, Kalani accounts for the subtraction sign.

"I am moving the opposite direction [because of the subtraction sign], so I would write down negative 5 here."

Pedagogical Goals

Procedure Only

- Stated rules clearly
- Expected students to practice rules and demonstrate their use

"They would all do that or they would get it wrong. I call my classroom "Smithsville," and you do it the Smiths way."

Pedagogical Goals

Procedure Only

- Stated rules clearly
- Expected students to practice rules and demonstrate their use (Smithsville)

Procedural With Conceptual

- Professed valuing *conceptual understanding* but were unable to explicate what that might entail.

"I want them to reason through it as much as they can. They need to know that there is some other reason [beyond the rule]."

"… just kind of picture it in their head."

"… have a good understanding of what positives and negatives mean."

Pedagogical Goals

No relationship was found among teachers between their
a) flexibility /robustness of reasoning and
b) stated pedagogical goals.

Pedagogical Goals

- Did we expect teachers to mention ways of reasoning as goals for instruction?
- No!
- Did we expect teachers to *apply* ways of reasoning?
- We did not know what to expect.
- Did they *apply* ways of reasoning?
- Yes!

Interpreting Student Thinking
Grade 4 -5 + -1 = □

- "Minus 5 plus minus 1 equals minus something …; 5 plus 1 equals 6, so this is minus 6 …. If you add these two together, it makes it farther from the positive numbers."

Interpreting Student Thinking
Grade 4 -5 + -3 = □

- **"Minus 5 minus minus 3 equals something. It would probably be something. It would probably be minus 2 because if you add … . If you use addition with this, it would be farther from the positive numbers, so if you do the opposite, it should be closer."**

Interpreting Student Thinking

Appreciating Understanding

$^1/_3$ Difficulty understanding student reasoning

Kalani: "I think he got confused. There's no context involved…So I don't see a clear understanding on his part at all." (referring to -5 + -1 = □)

"I don't quite understand him when he used the opposite. Opposite of what? …There is no context." (referring to -5 – -3 = □)

Interpreting Student Thinking

Appreciating Understanding

- $^1/_3$ **Difficulty understanding student reasoning**
- $^1/_3$ **Struggled appreciating nuances, either overgeneralizing or being overly critical.**

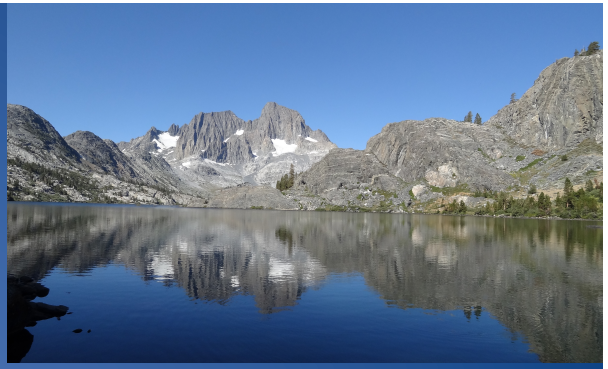
"He understands the rule that adding a negative and subtraction are exactly the same."

Interpreting Student Thinking

Appreciating Understanding

Correlation?

Teachers' Ways of Reasoning	No	**Pedagogical Goals**
Pedagogical Goals	Yes	**Interpreting Student Thinking**



Professional Noticing of Students' Mathematical Thinking

- Attending
- Interpreting
- Deciding how to respond

Specialized Content Knowledge

- What is specialized content knowledge (SCK)?

SCK is mathematical knowledge that teachers hold that they do not explicitly teach but that enables them to effectively teach their students.

- SMSs' knowledge of Andrew Task
- 7^{th}-grade teachers' knowledge of integer reasoning

Discussion

A Mathematical Road to SCK

A Road to SCK Through Children's Mathematical Thinking

Brian solves $-9 + x = -4$

"Because I saw the negative 4 and I know negative 9 is smaller than negative 4, so I knew it had to be positive. … So how do you go from 9 to 4? You have to subtract 5."

Although to us Brian seems to be using reasoning, he repeatedly asserts that he is using underlying rules.