Witches, Astrology, and Negative Numbers

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Project Z: Mapping Developmental Trajectories of Students' Conceptions of Integers

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"Above all, he [the student] must reject the definition still sometimes given of the quantity -*a*, that it is less than nothing. It is astonishing that the human intellect should ever have tolerated such an absurdity as the idea of a quantity less than nothing; above all, that the notion should have outlived the belief in judicial astrology and the existence of witches, either of which is ten thousand times more possible."

--Augustus De Morgan, 1898
On the Study and Difficulties of Mathematics, p. 72

Outline of Talk

- Project overview
- Negative-number challenges
- Historical perspective
- Ways of Reasoning
- Initial trends across grade bands
- Discussion/questions

Project Goal

- To describe K–12 students' conceptions of integers and operations with integers (+ and - only) and map possible learning trajectories.

Why Focus on Students' Mathematical Thinking?

- A large literature base surrounds students' mathematical thinking and ways of reasoning (e.g., Cognitively Guided Instruction; Karen Fuson and colleagues' work; and Purdue's Problem-Centered Mathematics Project).
- Children's mathematical thinking should inform instructional decisions when teachers use this information to productively support and extend children's reasoning (Black & Wiliam, 1998; Fennema et al., 1996; Vygotsky, 1998; Wiliam et al., 2004).

Why Focus on Students' Mathematical Thinking?

- Children often approach problems and reason about them differently than adults do.
- Consider the problem -2 + __ = 4. Solve it in two ways.
- Liberty is a 2nd grader who had heard of negative numbers and could add and subtract integers using counting strategies. Liberty answered 7 when solving this problem. How might she have arrived at that answer?

- Liberty, Grade 2, $-2 + \square = 4$

Project Overview

- Developed interview for grades K–12 and piloted more than 90 interviews
- Conducted 160 interviews—cross-sectional design
  - 40 from each of Grades 2, 4, 7, and 11 across 11 ethnically diverse school sites w/ varying API scores.
  - Problem-solving interviews lasted approximately 1.5 hours
  - Open number sentences, compare problems, story problems (majority of items were open number)
- Beginning to code and analyze data

What we know so far ...

- Young children have ideas about negative numbers that can be leveraged.
- Different ways of reasoning have both limitations and affordances.
- Understanding integers is complex! One way of reasoning is likely insufficient for making sense of integer operations.
- Similar to mathematicians during the historical development of integers, students, too, can use fundamental mathematical principles and formalisms to reason about negative numbers.

What’s so hard about integers anyway?

The Un-natural Nature ōf Z

- The existence of quantities less than nothing.
- Removing something from nothing or more than you have.
- Counterintuitive situations involving routine interpretations of addition and subtraction.

Quantities Less than Nothing?

- "A number tells you, like, how much of something it is. Negative numbers aren't really numbers; they just act like numbers .... I mean there is no negative 1 cube" (Rosie, 2nd grade).

- “Numbers that are less than nothing? Inconceivable! The next thing you’ll tell me is that there’s a witch outside with my horoscope” (Augustus De Morgan, as interpreted by Jessica).
- How can you have a negative number of monkeys? (Bhascara I, 7th century)
- How can you buy a negative amount of cloth from a merchant? (Chuquet, 1400s)

Taking away more than you have?

- Andrew, Gr 2,   3 – 5 = __

Something From Nothing?

- “Three minus 5 doesn’t make sense because 3 is LESS than 5.” (Niki, Gr1)
- “You may put a mark before 1, which it will obey: it submits to be taken away from another number greater than itself, but to attempt to take it away from the number less than itself is ridiculous. Yet this is attempted by algebraists who talk of numbers less than nothing ....”
- --William Frend, *The Principles of Algebra*, 1796

Something From Nothing?

- Seth, Gr 1,   3 – 5 = __

Something From Nothing?

- Seth keeps good company.

"I know people who cannot under-
stand that when you subtract four
from zero, what is left is zero."

--Blaise Pascal, *Pensées*, 17th century

$0 - 4 = 0$??

Addition Makes Smaller?

- No way!
    - Little Diophantus, 2nd grade, 6 + __ = 4

Addition Cannot Make Smaller

- [redacted] + 20 = 4 is “absurd” because the four units as the result of the summation “ought to be some number greater than 20.” (Diophantus, Grade ?, 3rd century)
- “4 + __ = 3 is not a real problem. It’s not true.” (Brad, Gr 1)

Lessons From History

- For these children and mathematicians alike, numbers "less than nothing" were a paradox.
- Mathematical formalisms helped to broaden mathematicians' ideas about number and encouraged the acceptance (finally!) of negative numbers. Why?
  - Negative numbers enabled mathematicians to solve algebraic equations that could not be solved otherwise.
  - The advent of abstract algebra enabled mathematicians to recognize that various number domains were possible, that including negatives afforded nice properties (e.g., inverses), and that different domains could have different properties.

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Students' Integer Reasoning

	Negative integers **not in** numeric domain	Negative integers **in** numeric domain
2nd grade	27	13
4th grade	13	27
7th grade	0	40
11th grade	0	40
	40	120

Students With Negative Integers

- Children who have negative integers in their conceptual domains often approach open number sentences using one of the following ways of reasoning:
  - Order-based reasoning
  - Magnitude-based reasoning
  - Computational reasoning
  - Limited ways of reasoning
  - Formal mathematical approach (e.g., logical necessity)
  - Other/unclear

Students With Negative Integers

- Order-based reasoning
  - Number line/motion, Grade 7,   -3 + 6 = __

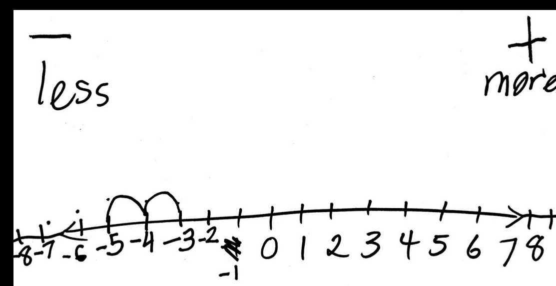
Students With Negative Integers

- Order-based reasoning
  - Count by ones, Liberty, Grade 2,   -2 + __ = 4

Predictions

- Rosie, a second grader, will use a number line and the idea of motion to try to solve each of the following problems:
  - __ + 5 = 3
  - -5 – 4 = __
  - 5 + __ = 2
  - 3 – __ = -2
- She correctly solves 3 of the 4. Predict which one she did not solve correctly.

- Rosie, Grade 2, Motion on a Number Line
- __ + 5 = 3, -5 – 4 = __, & 3 – __ = -2



- Rosie, Grade 2, Motion on a Number Line
- 5 + __ = 2

Next Steps

- Think about Rosie who solved _ + 5 = 3, 3 – __ = -2, and -5 – 4 = __ on the number line but could not solve 5 + ___ = 2 and insisted that to evaluate 6 + -2 is impossible (you did not watch this clip).

- Why did she struggle to solve 5 + __ = 2 and 6 + -2? What problem(s) might you pose next to extend her thinking?

What meaning is assigned to adding or subtracting a negative number?

What's the Difference?

- Rosie *could not* solve

$$5 + \square = 2$$

- but *could* solve

$$\square + 5 = 3$$

$$\boxed{\text{Starting Point}} + \boxed{\text{Change}} = \boxed{\text{Ending Point}}$$

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  - Formal mathematical approaches
  - Other/unclear

Students With Negative Integers

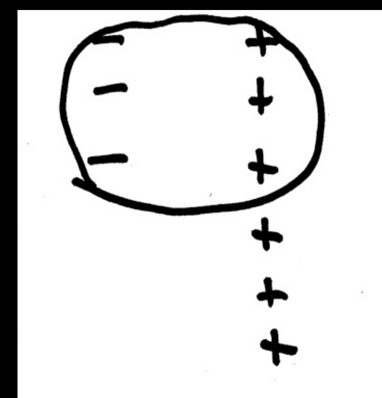
- Magnitude-based reasoning
    - Negatives like positives, Paulino, Grade 11,   -5 + -1

Students With Negative Integers

- Magnitude-based reasoning
  - Negatives like positives, Marlee, Grade 4, -5 – -3 =

Students With Negative Integers

- Magnitude-based reasoning
  - Chips/Inverses, version *a*, Grade 7,   -3 + 6



Students With Negative Integers

- Magnitude-based reasoning
  - Negs have magnitude (like a chips model) Grade 1,

$-8 - -1$

Students With Negative Integers

- Magnitude-based reasoning
  - Metaphor-owing, Grade 4, -3 + 6

Affordances and limitations of a magnitude-only approach to negative numbers

- James, Grade 1, Treats Negatives Like Positives
- -7 – __ = -5

- James, Grade 1, Treats Negatives Like Positives
- 1 + -2 = ___

Next Steps

- Think about James, who solved -5 + -2 and

-7 – __ = -5 but could not solve 1 + -2 = __
  or -1 + 4 = __.

- Why did James struggle with 1 + -2 and -1 + 4? What problem(s) might you pose next to extend his thinking?

Negatives & Positives As Animals in a Zoo

- If negatives are lions and positives are zebras, they must stay in their own cages.
- Of course they don't interact.
- For magnitude-only folks, the main issue is developing meaning for how to *blend* these numbers.

Students With Negative Integers

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    - Computational reasoning
    - Limited ways of reasoning
    - Formal mathematical approaches
    - Other/unclear

Students With Negative Integers

- Computational-based reasoning
  - Keep Change Change (KCC), Gr 11 & 7,
  - 6 – -2 = __ & 5 – __ = 8

- Computational approaches aren’t bad.
- But they can come with trade-offs.
- Although most students could correctly and accurately use rules and procedures, most could not justify them mathematically.

Students With Negative Integers

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  - Formal mathematical approaches
  - Other/unclear

Students With Negative Numbers

- Logical Necessity, Grade 1,   5 + -2 cf. -2 + 5

Students With Negative Integers

- Logical Necessity, Grade 11, 6 – -2 cf. 6 – 2

What we know so far ...

- Young children have ideas about negative numbers that can be leveraged.
- Different ways of reasoning have both limitations and affordances.
- Understanding integers is complex! One way of reasoning is likely insufficient for making sense of integer operations.
- Similar to mathematicians during the historical development of integers, students, too, can use fundamental mathematical principles and formalisms to reason about negative numbers.

What have we learned about integer reasoning while it develops over time?

Big Ideas

Prior to Formal Instruction

- Many young children have productive ideas about negative numbers.
- Children are sense makers and often engage in *doing* mathematics (Stein et al., 1996).
- Many children are consistent in their approaches to and reasoning about integer problems.
- In young children we see the emergence of reasoning about negative numbers in terms of what we call "logical necessity."

Big Ideas

After Formal Instruction

- Most middle school students...
  - invoke rules about how to add and subtract negative numbers.
  - struggle to explain the rules they invoke.
- Teachers' dilemmas include
  - how to support student reasoning in concert with developing efficiency
  - how to distinguish "rule following without understanding" from "reasoned" generalizations

Big Ideas

Long After Formal Instruction,

Successful Mathematics Students

- Most successful high school students invoke a variety of ways of reasoning to solve problems involving integer addition and subtraction.
- Most high school students struggle to reason about the equivalence between subtraction and adding the inverse.
- Many successful high school students have not developed a meaning for “–” as negation (or the opposite of).
- Successful high school students have many tools in their integers belt, yet still have room to grow!

Thank You

Discussion/Questions