Understanding Students' Pre- and Post-Instructional Conceptions of Integers and the Implications for Teacher Educators

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Solve Each in More Than One Way.

1) 3 – ___ = -6
2) -5 + -1 = ___
3) -8 – 3 = ___
4) 6 + ___ = 4
5) -5 – -3 = ___
6) 6 – -2 = ___
7) -8 – ___ = -2

Project Z: Mapping Developmental Trajectories of Students' Conceptions of Integers

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Project Overview

- Cross-Sectional
- Problem-Solving Interviews, duration 1–1.5 hours
- Open Number Sentences, Story Problems, Comparison Problems

Grade Level	Number of Students	Number of School Sites	Features

Project Overview

Grade Level	Number of Students	Number of School Sites	Features
2	40	4	9 classrooms, Pre instruction
4	40	4	8 classrooms, Pre instruction
7	40	4	11 classrooms, Post Instruction

Project Overview

Grade Level	Number of Students	Number of School Sites	Features
2	40	4	9 classrooms, Pre instruction
4	40	4	8 classrooms, Pre instruction
7	40	4	11 classrooms, Post instruction
11*	40	4	*Successful, ONLY Precalc/ Calc Classrooms

Session Goals

- Share Ways of Reasoning (WoR) and the Problem Types that evoke particular Ways of Reasoning.
- Characterize Ways of Reasoning preinstructionally and postinstructionally
- Discuss implications for teacher educators

Ways of Reasoning and Problem Types

	Way of Reasoning Evoked	Subcategory
1) 3 – ___ = -6		
2) -5 + -1 = ___		
3) -8 – 3 = ___		
4) 6 + ___ = 4		
5) -5 – -3 = ___		
6) 6 – -2 = ___		
7) -8 – ___ = -2		

$3 - \_\_ = -6$

Ways of Reasoning and Problem Types

	Way of Reasoning Evoked	Subcategory
1) 3 – ___ = -6		
2) -5 + -1 = ___		
3) -8 – 3 = ___		
4) 6 + ___ = 4		
5) -5 – -3 = ___		
6) 6 – -2 = ___		
7) -8 – ___ = -2		

$-8 - 3 = \_$

Ways of Reasoning and Problem Types

	Way of Reasoning Evoked	Subcategory
3 – ___ = -6	Order based	Number Line, Jumping to 0
-5 + -1 = ___		
-8 – 3 = ___	Order based	Number Line
6 + ___ = 4		
-5 – -3 = ___		
6 – -2 = ___		
-8 – ___ = -2		

-5 + -1 = __, Analogically Based, Negatives Like Positives

Ways of Reasoning and Problem Types

	Way of Reasoning Evoked	Subcategory
3 – ___ = -6	Order based	Number Line, Jumping to 0
-5 + -1 = ___		
-8 – 3 = ___	Order based	Number Line
6 + ___ = 4		
-5 – -3 = ___		
6 – -2 = ___		
-8 – ___ = -2		

-5 + -1 and -5 – -3, Analogically Based, Negatives Like Positives

Ways of Reasoning

	Way of Reasoning Evoked	Subcategory
3 – ___ = -6	Order based	Number Line, Jumping to 0
-5 + -1 = ___	Analogically based	Negatives Like Pos, Chips
-8 – 3 = ___	Order based	Number Line
6 + ___ = 4		
-5 – -3 = ___	Analogically based	Negatives Like Pos
6 – -2 = ___		
-8 – ___ = -2	Analogically based	Negs Like Pos

6 + _= 4, Formal Way of Reasoning, Infers Sign

Ways of Reasoning

	Way of Reasoning Evoked	Subcategory
3 – ___ = -6	Order based	Number Line, Jumping to 0
-5 + -1 = ___	Analogically based	Negatives Like Pos
-8 – 3 = ___	Order based	Number Line
6 + ___ = 4	Formal	Infers Sign
-5 – -3 = ___	Analogically based	Negs Like Pos,
6 – -2 = ___		
-8 – ___ = -2	Analogically based	Negs Like Pos

6 – -2, Computational

Computational, Equation

Ways of Reasoning

	Way of Reasoning Evoked	Subcategory
3 – ___ = -6	Order based	Number Line, Jumping to 0
-5 + -1 = ___	Analogically based	Negatives Like Pos
-8 – 3 = ___	Order based	Number Line
6 + ___ = 4	Formal	Infers Sign
-5 – -3 = ___	Analogically based OR Computational	Negs Like Pos, KCC
6 – -2 = ___	Computational	KCC
-8 – ___ = -2	Analogically based	Negs Like Pos

Ways of Reasoning

	Way of Reasoning Evoked	Subcategory
3 – ___ = -6	Order based	Number Line, Jumping to 0
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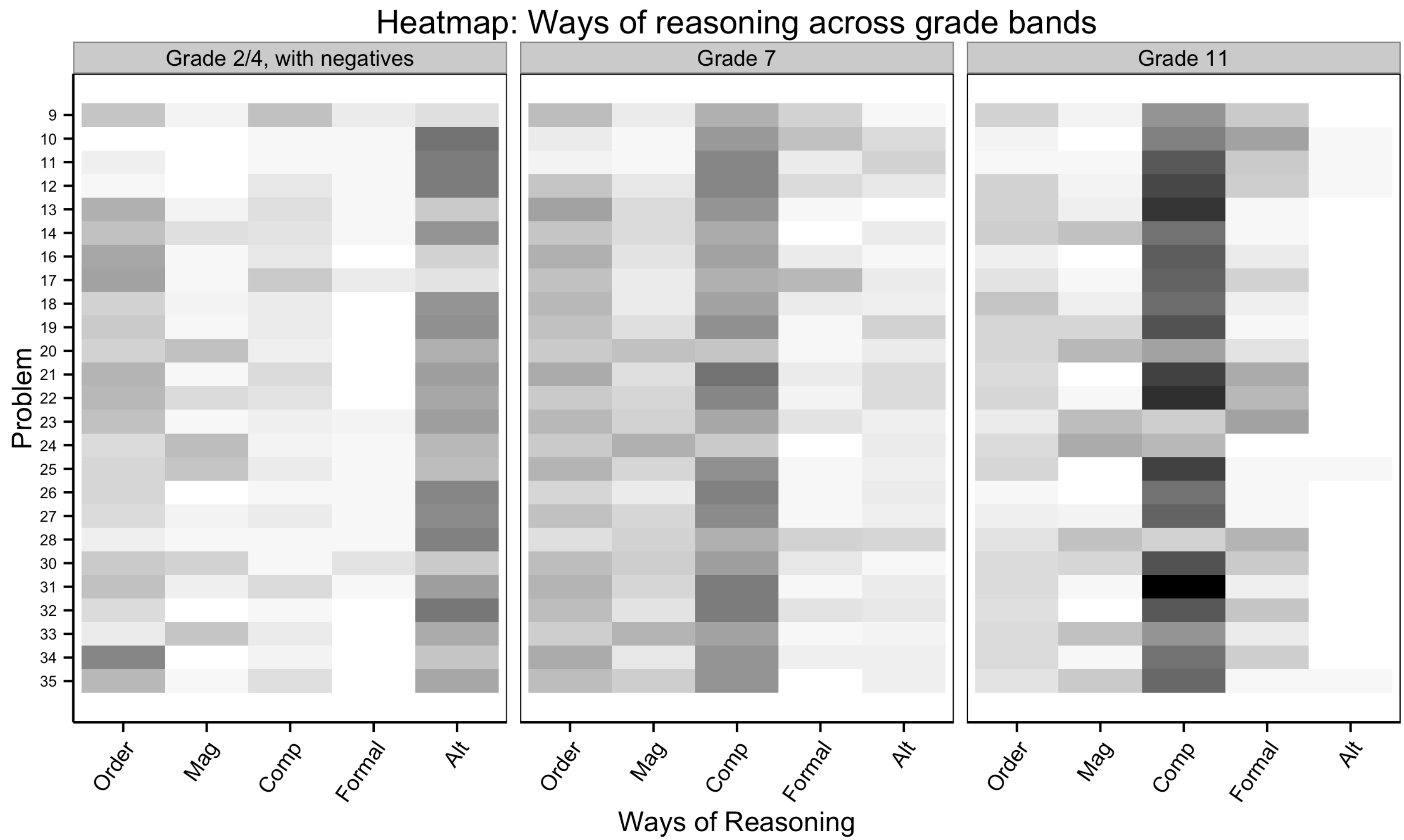
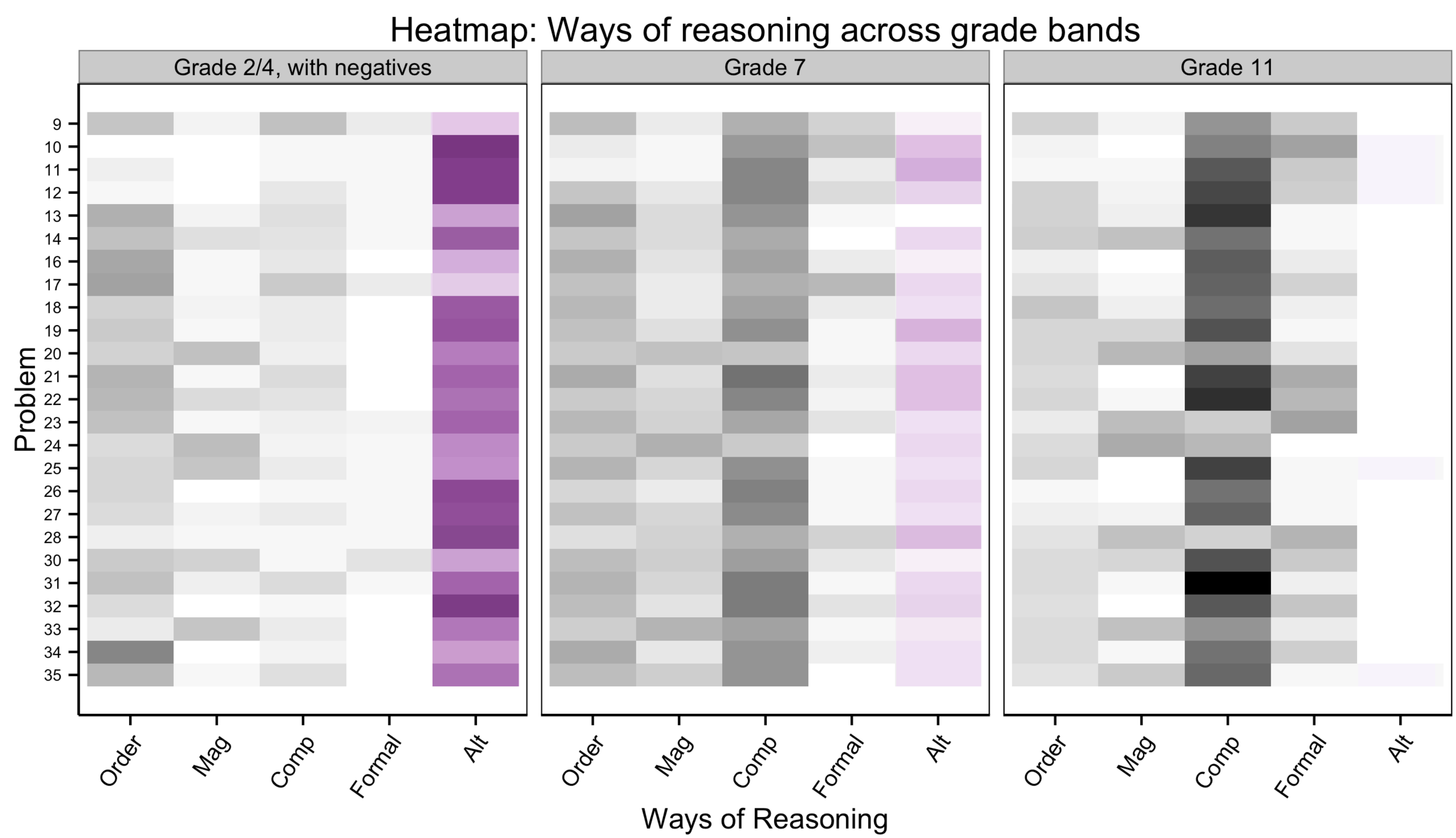
Ways of Reasoning

- Order based
- Analogically based
- Formal
- Computational
- Alternative

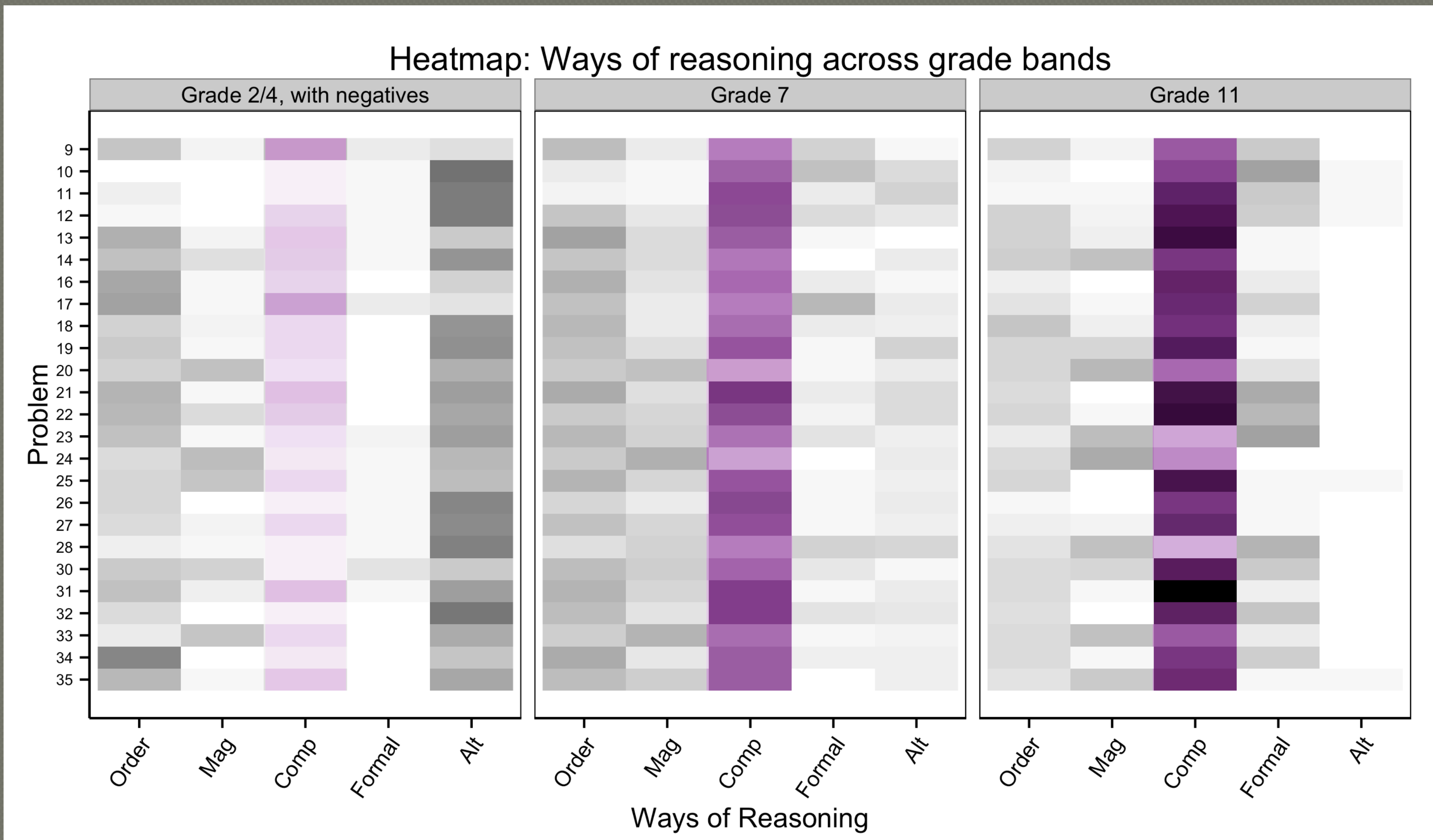
Ways of Reasoning, Heat Map



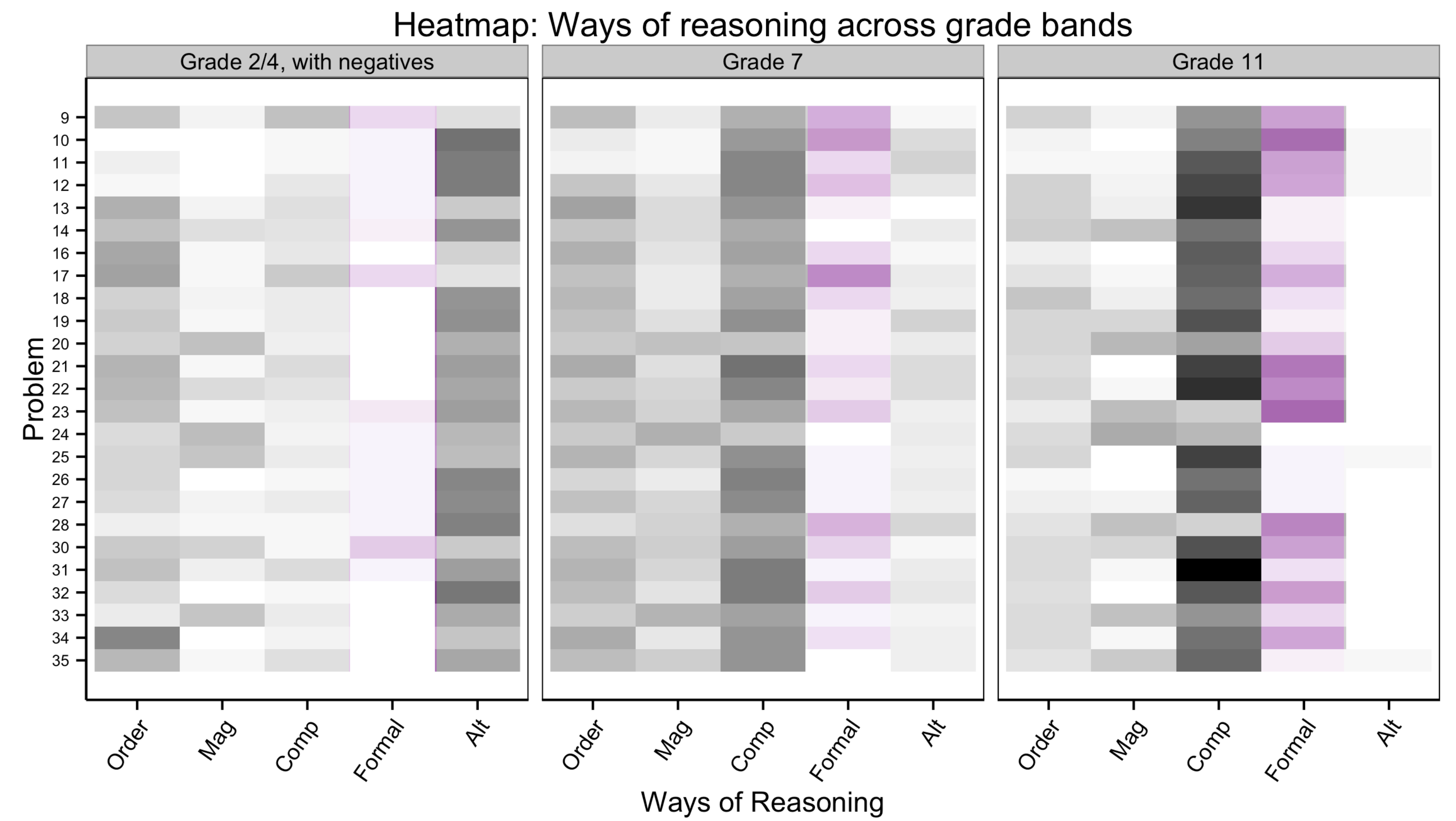
Use of Alternative Ways of Reasoning Across Grade Levels



Use of Computational Across Grade Levels

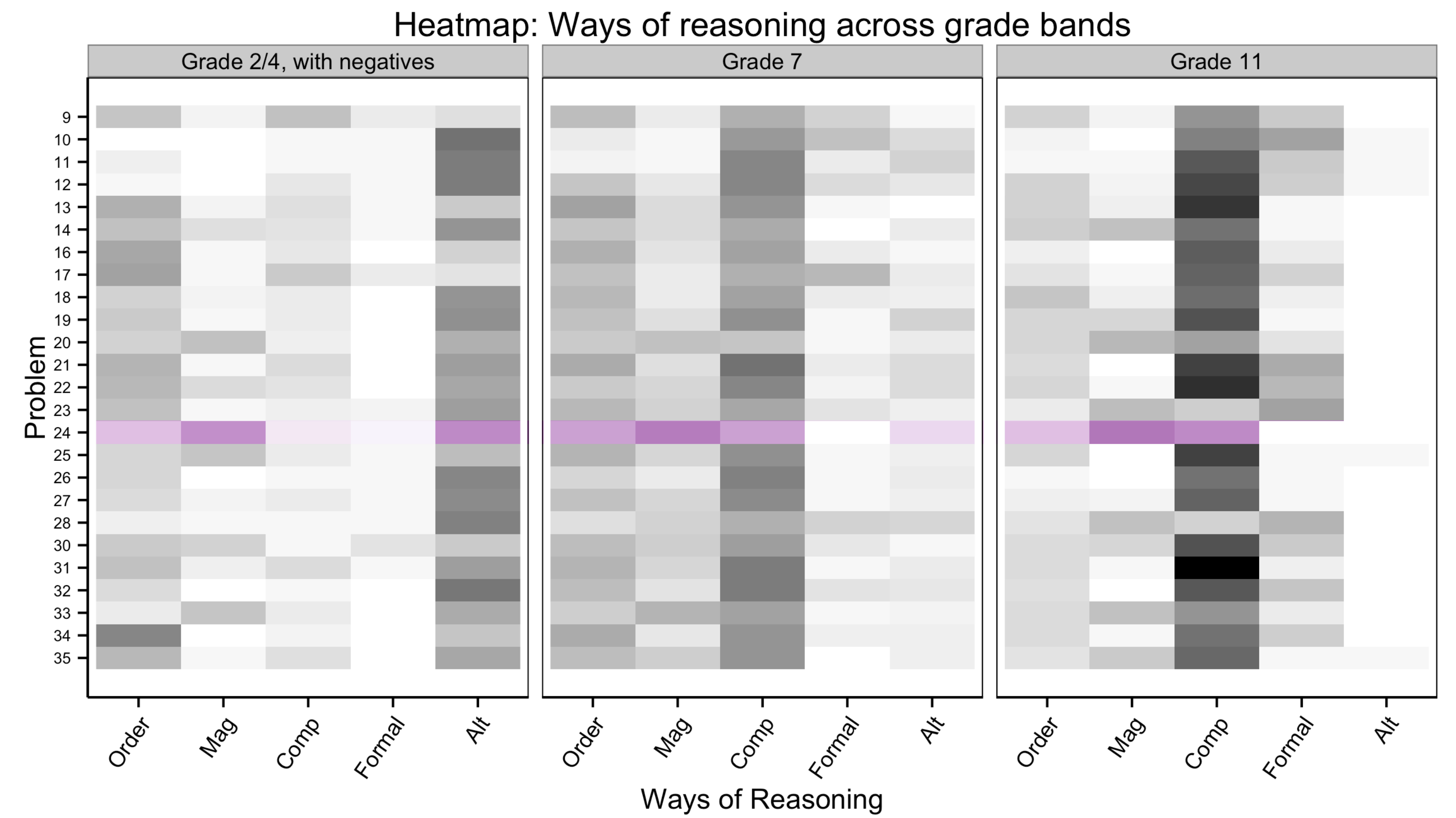


Use of Formal Ways of Reasoning Across Grade Levels



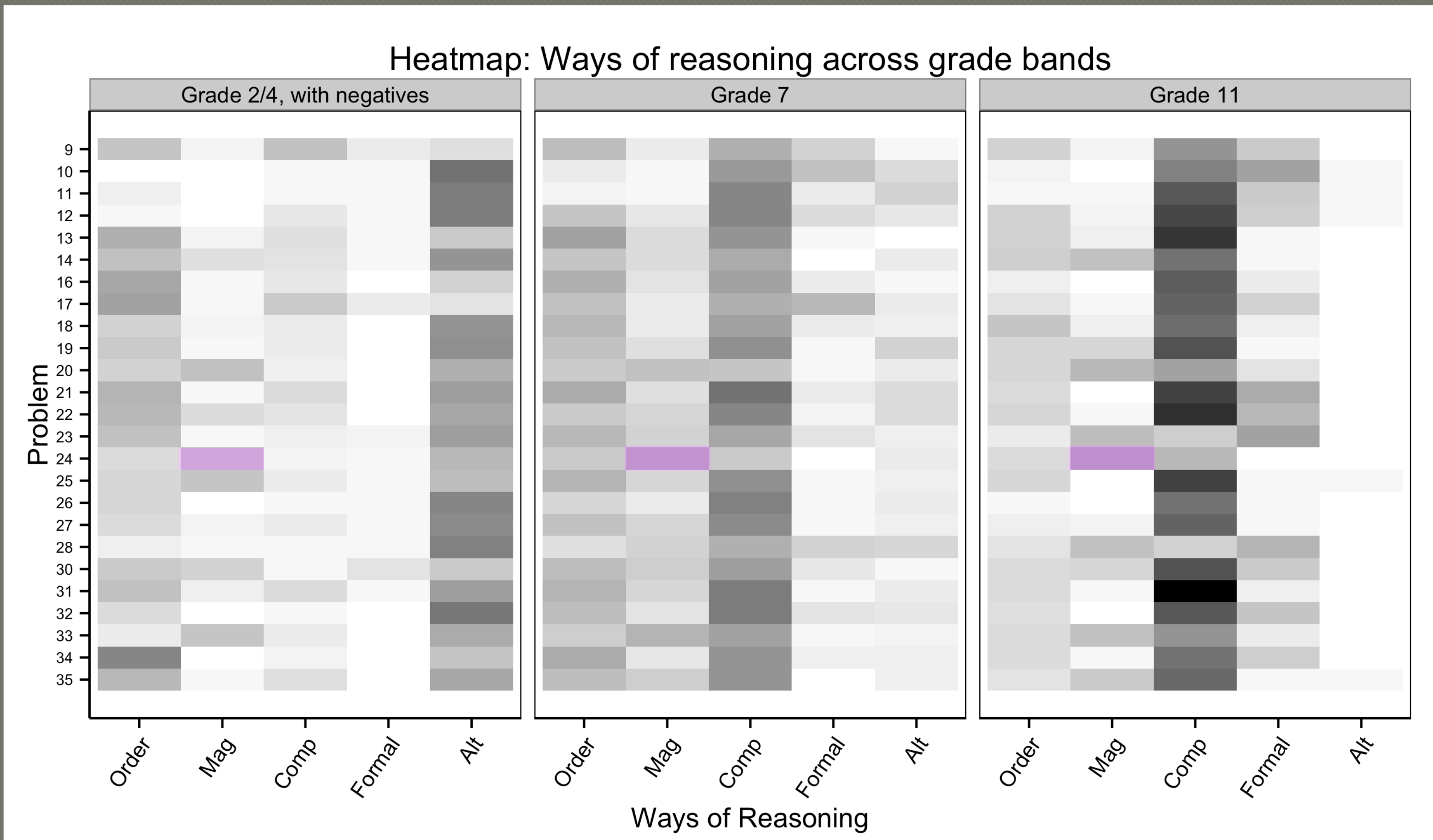
Analogically Based Example Across Grades

(Problem 24) -5 + -1 = ___

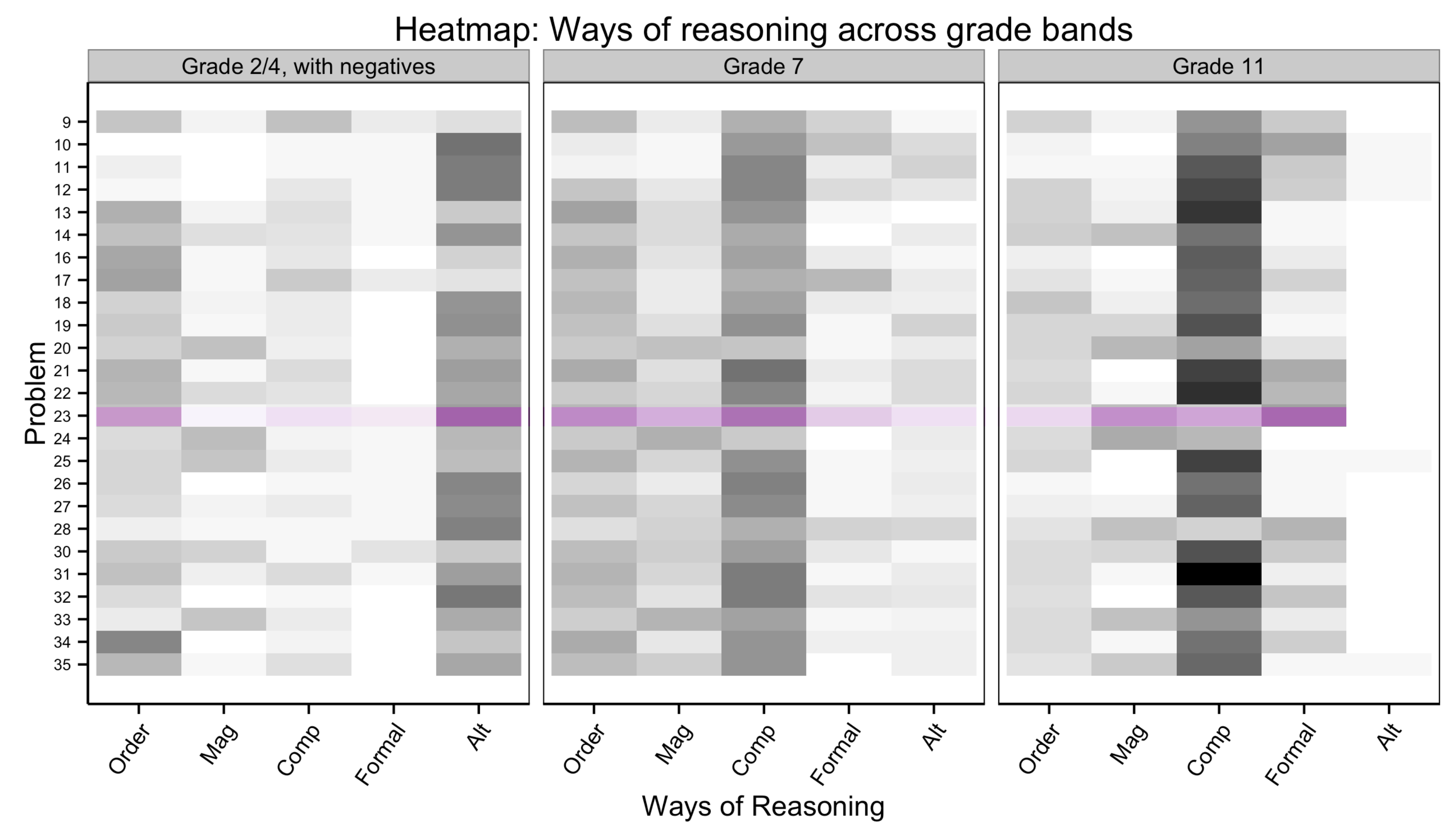


Analogically Based Example Across Grades

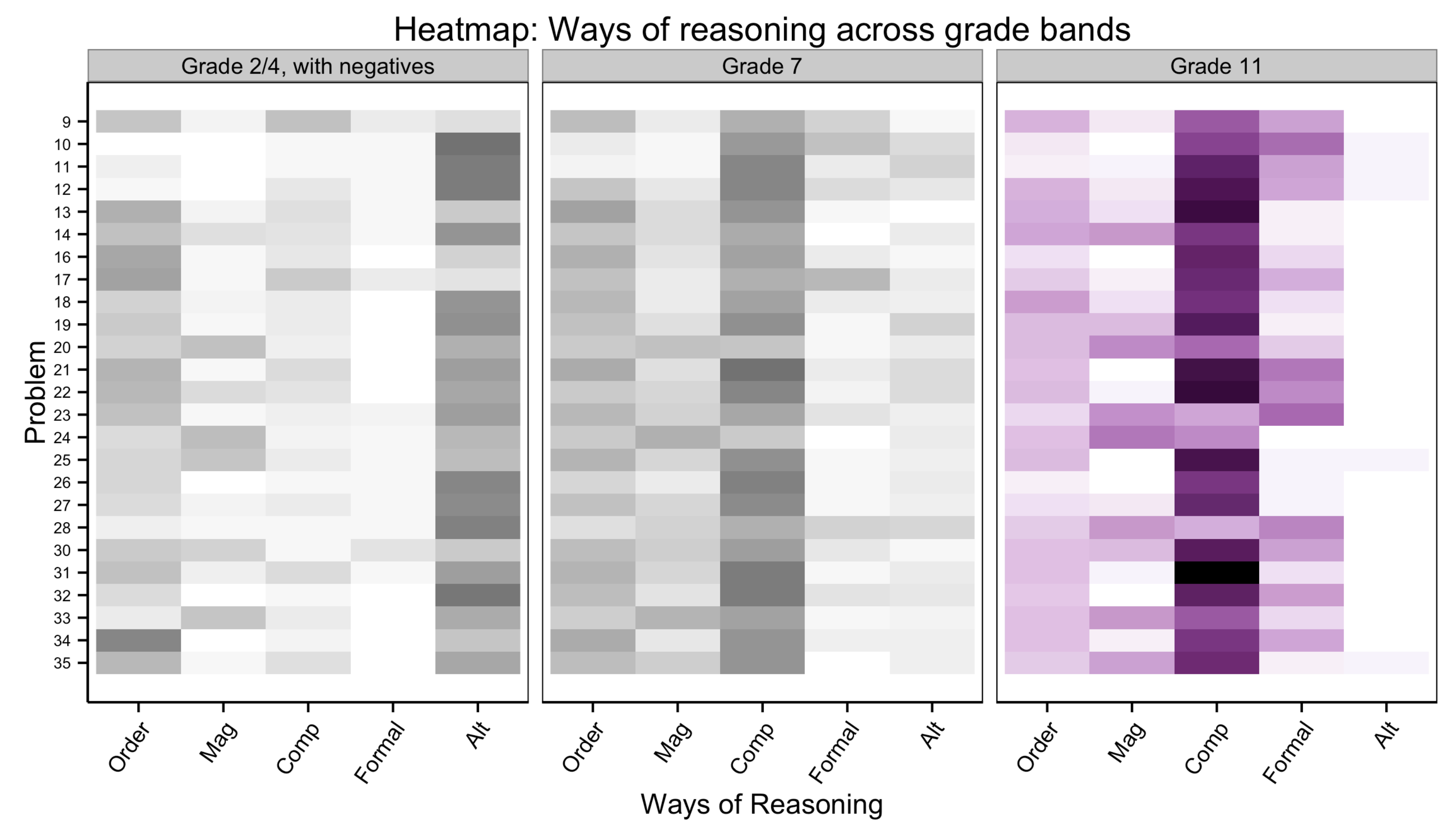
(Problem 24) -5 + -1 = ___



Variety of Ways of Reasoning (Problem 23) -8 + __ = 0



Ways of Reasoning Successful 11th Graders



Successful 11th Graders

- Almost all correct answers (98.5%)
- Extensive use of Computational strategies (75%)
  - Most unable to justify those procedures

Keep, Change, Change

Successful 11th Graders

- Almost all correct answers (98.5%)
- Extensive use of Computational strategies (75%)
  - Most unable to justify those procedures
- Use of a variety of other strategies (60%)
  - Often in combination with Computational
  - Formal reasoning
    - Infers Sign

Infers Sign

2nd and 4th Graders Without Negative Numbers

- *Computational* strategies (13%)
  - Treating 3 – 5 as 5 – 3
  - Treating minus signs as indicating subtraction (e.g., -3 + 6 means subtract 3 and then add 6)
- *Alternative* strategies (93%)
  - Ignores Sign (e.g., -5 + -1 = 6)
  - Conceptions such as *Addition Makes Bigger* (AMB) / *Subtraction Makes Smaller* (SMS) *Limits*

Addition Makes Bigger

2^{nd} and 4^{th} Graders With Negative Numbers

- Overall percentage correct is 35.5%, 3%–87% correct.
- Difficult problems?
  - Counterintuitive problems
    6 + __ = 4, 5 – __ = 8, -3 – __ = 2
  - Combining a negative and a positive number
    6 – -2 = __, -2 – 7 = __, 6 + -3 = __
- Easy problems?
  - All negative numbers, -5 + -1 = ___
  - Crossed zero with no "double signs"
    3 – __ = -6, -3 + 6 = __.

Surprising Findings

- Four easiest problems for 2nd and 4th graders who have negative numbers

  -5 + -1 = __, -5 – -3 = __,

  __ + -2 = -10, and -5 – -5 = __

  (all ≈ 75% correct).

- On these problems except on __ + -2 = -10, 2nd and 4th graders outperformed 7th graders.
- Why?

Grades 2 & 4 Findings

- Students used a specific Analogical strategy, *Negatives Like Positives,* to correctly solve problems like -5 + -1 = __, -5 – -3 = __, __ + -2 = -10, -8 – __ = -2, and -5 + __ = -8.
- Students who appropriately used *Negatives Like Positives* more than once also used *Can't Mix* on one or more of these problems: 6 – -2 = __, 6 + -3 = __, -8 – 3 = __, -3 + 6 = __.
- This pattern is suggestive of a broader interaction between problem types and strategies.

Animals in the Zoo

- James, -7 – __ = -5
- James used the strategy of *Treating Negatives Like Positives to solve -7 – __ = -5.*
- *James knew that for "real" numbers, 7 – 2 = 5, so he made a conjecture that subtraction with negative numbers worked similarly. If he added a negative sign to each number, then -7 – -2 should be -5.*

Animals in the Zoo

- James, 1 + -2 = __
- For 1 + -2, James struggled because he did not have a model for how to combine a negative and a "real" number. He clearly explained that he could solve a problem like -5 + -2 but he could not solve this problem. The difference is that the former involved all negative quantities but this problem involved a negative and a positive quantity. Thus, he concluded that when you add a "real" number and a negative number, the result is not a number at all!

7^{th}-Grade Findings

- Overall percentage correct is 74% (range 50%–90%).
- What was difficult?
  - "Double" signs (add or subtract a negative)
  - Counterintuitive problems (5 – __ = 8)
- What was easy?
  - Problems conducive to use of multiple approaches and ways of reasoning (no obvious problem type).

7th-Grade Findings

- 7th graders used more computational strategies.
- 7th graders used all Ways of Reasoning, often using strategies that are not the result of instruction (*Neg as Subt, Jump to Zero*, etc.)
- Within Order, *Number line/motion* was the most used strategy and was invoked on problems like -3 + 6 = __, 3 – __ = -6, and -2 + __ = 4.
- *Negatives Like Positives* and *Chips* were the most popular Analogically based strategies. Students tend to use both correctly for problems like -5 + -1 = ___ , -5 + __ = -8, and 3 + __ = 0.

Computational WoR in Grade 7

- The most commonly used computational strategies were, in general, used correctly: *KCC, Negative as Subtractive*, and *Commutativity*.
- A subset of the Computational strategies were more procedural or rule-based and were used on one third of the open number sentences: *KCC, Same-Sign/Diff-Sign Rules, Equation*, and so on. Although students tended to use KCC correctly (95% correct), the remaining procedural strategies were used correctly only 50% of the time. Some of these procedural strategies consistently led to incorrect answers (e.g., *Mult sign rules,* and *Nonequivalent Transformation*).

Nonequivalent Transformation

Many Middle-School Students Do Not Understand.

- Valentina and friends … Does 6 – -2 = 6 + + 2?

- To a student who answered that the two expressions are not equal, the interviewer said, "That's kind of crazy that you are allowed to change the problem and it gives you a different answer."
- The student responded, "That's math."

- We wonder what sense students are making of some of the computational strategies—in particular, the effects of transforming the original problem in ways that preserved the solution set but changed the problem itself.

Take-Away Points

- Ways of Reasoning interact with problem types, and WoR change over grade levels.
- Young children can reason productively about negative numbers before instruction.
  - Use of Order-based Ways of Reasoning and Analogically based Ways of Reasoning before instruction.
- Flexibility—using a variety of Ways of Reasoning coupled with the ability to **select** a way of reasoning depending on problem type (signs of numbers, operation, location of the unknown) is a key feature of robust integer sense.

Implications

What are the implications of this work?

- Encourage students to engage with negative numbers earlier than CCSS suggests.
- Provide open number sentences and ask students to discuss how they reasoned.
- Select open number sentences that evoke certain ways of reasoning.
- Given the value of flexibility, promote the use of a variety of ways of reasoning (rather than a single favorite model or variety of **models**).

Magnitude & Order in Grade 7

- *Negatives Like Positives* (70 instances) and *Chips* (68 instances) were the most popular magnitude-based strategies. Students tend to use both strategies correctly on problems like -5 + -1 = ___ , - 5 + __ = -8, and 3 + __ = 0.
- *Number line/motion* is the most popular order-based strategy (73%) with *Counting* strategies and *Jumping to Zero* occurring infrequently. Problems like -3 + 6 = __, 3 – __ = -6, and -2 + __ = 4 encourage order-based WoR.

Alternative & Formal in Grade 7

- Alternative ways of reasoning decreased in Grade 7, but they occurred in about 25% of the responses to 5 – __ = 8 and -9 + __ = -4 (most often as *Addition Cannot Make Smaller* & *Subtraction Cannot Make Larger*). When these strategies were invoked, students tended to give incorrect answers.
- Formal WoR occurred less often but resulted in correct responses. When students used Formal WoR, they almost always used *Infers Sign* (77% of Formal strategies). *Logical Necessity* is rarely invoked by 7th graders.

Ways of Reasoning: Percentage Use by Grade Level

Ways of Reasoning	2nd /4th Graders w/o Negatives	2nd /4th Graders With Negatives	7th Graders	11th Graders
Order	0%			19%
Analogical	0%			16%
Formal	0%			24%
Computational	13%			75%
Alternative	93%			<1%

Because students can use more than one way of reasoning to solve a problem, column percentage sums are larger than 100%.

Ways of Reasoning: Percentage Use by Grade Level

Ways of Reasoning	2nd /4th Graders w/o Negatives	2nd /4th Graders With Negatives	7th Graders	11th Graders
Order	0%			
Analogical	0%			
Formal	0%			
Computational	13%			
Alternative	93%			

Because students can use more than one way of reasoning to solve a problem, column percentage sums are larger than 100%.