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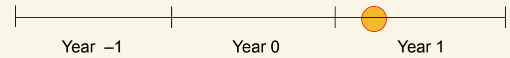
Project Z
Ways of Reasoning About Integers: Order, Magnitude and Formalisms

3-Year Project (2009–2012)



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"There are other numbers under zero. I know about them, but I don't know what they're called." (Lucy, 1st grade)
"Negative numbers aren't really numbers. They're just acting like other numbers." (Violet, 2nd grade)

Research Questions

1. What are Grades 2, 4, 7 and 11 students' conceptions of integers and operations on integers? In particular, how do Grades 2 & 4 children who have integers in their numeric domains reason about integers **prior** to formal, school-based instruction?
2. What are possible trajectories of students' ways of reasoning about integers?

Why Study Integers?

- When compared with the literature on rational numbers or place value, literature on students' understanding of integers is relatively sparse (NRC, 2001).
- Students often have great difficulty operating on integers, and those difficulties appear to be robust (see, e.g., Thomaidis & Tzanakis, 2007; Vlassis, 2002). Even those who have completed algebra courses are challenged by problems with negative numbers (Reck & Mora, 2004; Vlassis, 2002).
- Integers mark a transition from arithmetic to algebra because of their abstract nature (negative numbers have no concrete embodiments) and because students must understand fundamental algebraic principles, for example, using additive inverses, which first come into play with the introduction of integers.
- Difficulties in algebra have been linked to a lack of integer understanding (see e.g. Moses, 1989).

Methods

- Developed integers interview appropriate for Grades 1-12 students and piloted over 90 interviews.
- Conducted 160 problem solving interviews using a cross-sectional design.

  40 from each of grades 2, 4, 7 and 11 across eleven ethnically diverse schools with varying API scores in San Diego County.

  Problem-solving interviews lasted about 1.5 hours

  Interview tasks included open number sentences, compare problems, and context problems—majority of questions were open number sentences.
- Developed a coding scheme using constant-comparative method for interview tasks.

  Coding scheme identifies 5 broad categories of students' integer reasoning: ordinal reasoning, magnitude-based reasoning, alternate/limited reasoning, computationally-based reasoning, and formal-mathematical reasoning
- This study uses data from pilot interviews with 47 students in grades 1-4 who had not received any school-based integer instruction.
- Use composite case studies as exemplars of three ways of reasoning about integers: order, magnitude, and formalisms

Interview Tasks

Open Number Sentences	Comparison Tasks	Context Problems
Problems of the form: $6 - 11 = \square$ $12 + \square = 4$ $5 - \square = 8$ $-5 - -2 = \square$ $\square - 6 = -3$ $-9 + \square = -4$	For each pair, circle the larger, write "=" if the two quantities are equal, or write "?" if there is not enough information to determine which is larger. a) 3 -7 b) 0 -9 c) -5 -6 d) -100 -5 e) - -3 -3 f) $\square$ $-\square$	Yesterday you borrowed \$8 from your friend to buy a school t-shirt. Today you borrowed another \$5 from the same friend. Does your friend owe you money, do you owe your friend money, or is it some other situation? Can you write an equation or number sentence that describes this story and explain how it relates to the story?

Three Ways of Reasoning About Integers

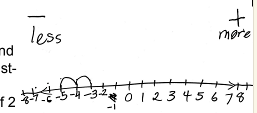
Violet: Using Order to Reason About Integers

In this way of reasoning the child imposes an ordering on $\boxed{\mathbb{Z}}$, and uses the positional (or ordinal) nature of numbers to solve problems. Strategies include counting, use of number line, and the contexts of motion/movement.

Counting Strategy $-9 + 5 = \square$

"Negative 9 plus 1 (puts up a finger) would be negative 8. Plus 2 is negative 7 (puts up 2nd finger), plus 3 is negative, negative 6 (3rd finger), plus four is negative 5 (4th finger), and plus five is negative 4 (5th finger)."

Number Line $\square + 5 = 3$



Violet used trial and error, starting at -5 on the number line and counting up 5 places, one-by-one, to end at 0. She then adjusted and started at –4, counting up 5 places and ending at 1. After starting at -3 and, again, ending at the wrong number of 2 she answered, "Negative 2," and then counted up 5 places from –2, verifying she ended at 3.

Jumping to Zero $3 - 5 = \square$

"Well my brain was thinking that 3 minus 3 is zero. And that 0 minus 2, because 3 plus 2 is 5, um, that must be negative 2." This strategy, *Jumping to Zero,* is essentially a more efficient way of counting. Much like children use decade numbers in incrementing strategies to solve problems involving regrouping (e.g., jumping to 30 to solve 33 – 5), Violet used the "friendly number" of 0. Only here, when decomposing 5 into 3 and 2 she was crossing zero, not 30

Jacob: A Magnitude-based Way of Reasoning About Integers

This way of reasoning is characterized by relating numbers to a countable amount or quantity. It is tied to ideas about cardinality and views a number as having magnitude.

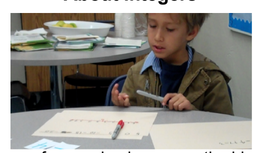
Treating Negatives Like Positives
$-7 - \square = -5$

Jacob answered -2 for this problem and explained, "Well for this one I need little cubes. It would be like real numbers but you just add the minus sign. You just do seven plus, well actually, seven minus two equals five. That's the answer for real numbers, so I just added a negative to all of them and there is my answer."

Negatives as Countable Objects
$-8 - -1 = \square$

In this problem, Jacob treated -8 as eight negative 1s or as equivalent to -1 + -1 + -1 + -1 + -1 + -1 + -1 + -1. Jacob explained, "You take away a negative because there is eight negative 1s altogether, so that would make negative 8. And you take away just 1 of it, and now it's -7."

Ryan: A Formal Way of Reasoning About Integers



This way of reasoning leverages the ideas of structural similarity, well-defined expressions, and fundamental mathematical principles (e.g., commutativity). It includes generalizing beyond a specific case by making a comparison to another, known, problem and appropriately adjusting one's heuristic so the logic of the approach is consistent.

Fundamental Properties—Commutative Property
$5 + -2 = \square$

Ryan answered, "Three. Because it's pretty much the same thing (points to $-2 + 5 = \square$ which he solved earlier). Five plus negative 2 and negative 2 plus 5—if you add the same things and you just say 5 first and [negative] 2 second, it's still the same thing. ...You always add the same things together." Ryan invoked a fundamental principle of mathematics, the commutative property, to reason about a possible meaning for adding a negative number. He assumed that all numbers, even negative numbers, obey the commutative property; consequently, his answer had to be 3.

Logical Necessity
$-5 - -3 = \square$

Before solving this problem, Ryan had solved $-5 + -3 = \square$, answering -8. "-5 – -3 … would probably be … minus two [another term for -2] because if you use addition with this (pointing to -5 – -3), it would be farther from positive numbers. So if you do the opposite, it should be closer. Here, Ryan compared the operations of subtraction and addition, holding the numbers -5 and -3 constant. He reasoned that if addition moves one further from the positive numbers, then its opposite, subtraction, should move one closer to the positive numbers. In other words, addition and subtraction cannot behave in the same way; they must be well-defined operations yielding unique values when given the same input.

Findings

- Young children have **productive** ideas about negative numbers that can be leveraged.
- Different ways of reasoning have both limitations and affordances.
- Understanding integers is complex! One way of reasoning is likely insufficient for making sense of integer operations.
- Similar to the historical development of integers, children, too, can use fundamental mathematical principles and formalisms to reason about negative numbers.
- Children are sense makers and engage in doing mathematics (Stein et al., 1996).