INTEGERS: HISTORY, TEXTBOOK APPROACHES, AND CHILDREN'S PRODUCTIVE MATHEMATICAL INTUITIONS

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We juxtapose the history of integers with current pedagogical approaches. We also present findings concerning children's reasoning about negative numbers prior to instruction. We see alignment between children's intuitions and the avenues that afforded progress for mathematicians, whereas the textbook approaches tend to run counter to lessons learned from history.

Introduction

The introduction of the notion of signed numbers is a critical point in children's learning of mathematics. As with fractions, the advent of integers widens the domain of numbers in children's mathematical worlds and, as such, presents notorious difficulties. On the one hand, some of the challenges that children experience in coming to grips with the notion of negative numbers parallel those that mathematicians encountered historically. On the other hand, the experience of most children today as they learn about integers is quite distinct from that of mathematicians of old. After all, at the time Diophantus was denying the possibility of negative solutions to linear equations—a position that was perfectly reasonable on the basis of his conceptions of number—he had no teacher or textbook telling him otherwise (Gallardo, 2002; Henley, 1999).

We looked to the history of integers to guide our review of current textbook approaches to integer instruction. The juxtaposition of these two bodies of evidence raises questions about the utility of prevalent textbook approaches. At the same time, we have seen children in interviews engage productively with ideas concerning positive and negative numbers prior to formal instruction. These findings show potentially fruitful alternative directions for textbook instruction concerning integers.

Theoretical Framework: A Historical Perspective

Before there were positive and negative numbers, there were simply numbers. The notion of *positive* is meaningful only in contrast with *negative*. Thus, the history of integers is largely the history of negative numbers. Negative numbers arose from the operation of subtraction, and they came about in at least two ways:

1. There is sense in which a subtrahend can be regarded as a *subtractive number* (Henley, 1999), which is to say that the subtraction operation is subsumed into the quality of the number. For example, instead of interpreting the expression 9 – 5 as meaning that we subtract 5 from 9, we might instead think in terms of 9 and "subtract 5" being combined additively. Under this interpretation, "subtract 5" becomes an entity that we can think about, even in the absence of any particular minuend. We can reason about -5 without

having a notion of this entity as a negative number, but simply as an amount to be taken away. Thus, 3 – 8 = -5 because we still must take 5 more away.

2. The natural numbers are not closed under subtraction. We must either restrict subtraction to cases in which the minuend is greater than or equal to the subtrahend, or we must confront situations in which the subtrahend is greater.

Negative numbers, or at least their ancestors, arose for mathematicians in these ways. For centuries, mathematicians struggled to make sense of such numbers by relating them to quantities. Some real-world situations were amenable to interpretation in terms of positives and negatives, whereas in other cases negatives were nonsensical (Colebrooke, 1817; Henley, 1999).

The history of negatives is a history of progress in the face of resistance. The mathematical machinery continued to be developed even as mathematicians held negatives in dubious regard, and some objected vehemently to their use. Resistance derived from legitimate paradoxes: How is it possible to take something from nothing? How could -7 be "smaller" than 3, let alone 0? Such questions plagued great thinkers and stymied progress (Hefendehl-Hebeker, 1991). Nonetheless, over time, more and more mathematicians used and came to embrace negatives because of their usefulness. The resistance to negatives stemmed from attachment to concrete or quantitative interpretations of number. Acceptance followed from an alternative, purely mathematical stance: Negative numbers were legitimate because they solved uniquely mathematical problems, not because they were necessary for solving problems concerning money, elevation, temperature, or any such thing and not because they made sense in all the same ways that regular (nonnegative) numbers made sense. Ultimately, it was the advent of abstract algebra that led the mathematical community to fully accept negatives as numbers, on par with positives. The integers came to be recognized as a domain, distinct from other domains such as the natural numbers, and it was unproblematic for different domains to have different properties (Henley, 1999).

Children's Introduction to Integers

The extension of the natural numbers poses challenges for both teachers and students. Children are expected to expand their mathematical worlds to include negative integers and (nonnegative) rational numbers, on the way to real numbers (Bruno & Martinón, 1999). These extensions challenge students' previous conceptions, which often involve overgeneralizations of their experiences with the natural numbers. There is a wealth of research concerning teaching and learning in the rational number domain (e.g., Empson & Levi, 2011; Fosnot & Dolk, 2002; Lamon, 1999; Sowder, 1995). However, few reports exist that address the challenges of students' learning about integers (Kilpatrick, Swafford, & Findell, 2001).

Integers and integer operations present conceptual difficulties for students (Janvier, 1983; Vlassis, 2004). These difficulties can be appreciated in light of the history of mathematics, wherein mathematicians struggled with counterintuitive notions associated with negative numbers (Gallardo, 2002; Henley, 1999; Thomaidis & Tzanakis, 2007). Mathematicians of old struggled with the distinction between magnitude and order that arises with negatives: There is a sense in which -7 is more than 3; however, it comes before 3 on the number line, and we use the term *less than* to refer to this relationship. Children encounter these same issues when introduced to negative numbers. However, our world has changed such that mathematically literate adults today can take negative numbers for granted. Thus, teachers may treat the introduction of negatives matter of factly, whereas for children it involves a conceptual revolution.

It was only after considerable conceptual struggles that great mathematicians such as Descartes and Newton came to accept negatives as numbers. It is no wonder, then, that children would face difficulties in making sense of negative numbers and operations involving them. At the same time, however, researchers have found children to be capable of reasoning about integers in relatively sophisticated ways, even in the lower elementary grades (Behrend & Mohs, 2006; Bishop, Lamb, Philipp, Schappelle, & Whitacre, 2011; Hativa & Cohen, 1995; Wilcox, 2008). For example, Bishop et al. found that first graders who had never heard of negative numbers nonetheless began to invent and reason productively about them in the contexts of playing a number-line game and solving open number sentences.

Methods

There are three branches of this study: historical review, textbook analysis, and analysis of children's thinking. In this section, we discuss our methods of data collection and analysis with respect to each of these three branches.

Historical Review

We reviewed sources concerning the history of mathematics with a focus on ideas associated with negative (and positive) numbers, especially the integers. We identified in the literature aspects of the extension to negatives that troubled mathematicians, as well as trends and insights that led to the acceptance of negatives as numbers. These findings further informed our analysis by providing a lens through which both the textbook approaches and children's reasoning could be interpreted.

Textbook Analysis

We analyzed the treatment of integer topics in 18 fifth- and sixth-grade mathematics textbooks currently adopted by the State of California. Our methods of analysis were both quantitative and qualitative. We counted the numbers of textbooks employing particular approaches or types of problems, so that descriptive statistics would provide a sense of the frequency of these approaches. We focused our analysis on those chapters/sections of textbooks devoted to integers directly—those lessons concerning what integers are and how to operate with them. Our qualitative analysis of the textbook content began with a process of open coding (Strauss & Corbin, 1998). We used principles of grounded theory (primarily the constant comparative method) to identify emergent, distinguishing themes and features of how integers are presented in textbooks (Strauss & Corbin, 1998). We balanced emergent codes from the data with historical and theoretical findings from the literature. We report on trends seen across the sample of textbooks and highlight pedagogical issues that stand out to us. In particular, we examine these textbook approaches through a lens informed by the history of integers.

Analysis of Children's Thinking

We conducted interviews with 55 elementary school students from Texas and California. Twenty-nine of these children were in grades K–2 at the time of the interviews; the rest were in grades 3–5. The responses that we discuss in this paper were to open number sentences, such as the problem $3 - 5 = \square$. These were used with interview participants at all grade levels because, whether or not K–5 children have been introduced to negative numbers, the question is at least sensible. They are familiar with the natural numbers and with subtraction. Those children who did have some familiarity with negative numbers were posed problems that explicitly involved

negatives, such as -5 + -1 = ☐. The interviews themselves were conducted at the children's school sites, during the school day. They lasted between 30 and 50 minutes and were videotaped.

Our analysis of the videotaped interviews was qualitative, focusing on children's solution strategies as well as their underlying ways of reasoning. Through constant comparative analysis, we identified various themes concerning students' reasoning about integers. We then related these to the history of mathematics, identifying parallels in the form of obstacles to progress in dealing with negative numbers as well as ways of thinking that afforded innovations. In this short paper, we highlight certain productive ways of reasoning.

Results: Textbook Approaches to Integer Instruction

A hallmark of current textbook approaches to the teaching of integers is the use of contexts and models. The most typical contexts used in textbook presentations of the integers themselves or of addition or subtraction involving integers are the following: money (used in 94% of the textbooks sampled), elevation (89%), temperature (89%), and movement forward and back (61%). The typical models employed in instruction involve (a) number lines and (b) colored chips (or, in some cases, "charged particles") that are used to represent positive ones and negative ones. An example of the use of a money context would be a person's bank-account balance and how it is affected by one or more transactions. The following story problem and example solution appear in the integer-addition section of a sixth-grade textbook:

> The Debate Club's income from a car wash was $300, including tips. Supply expenses were $25. Use integer addition to find the club's total profit or loss.

300 + (-25)	*Use negative for the expenses.*
300 – 25	*Find the difference of the absolute values.*
275	*The answer is positive.*

The club earned $275. (Burger, Bennett, Chard, Jackson, & Kennedy, 2007)

Perhaps the most poignant theme concerning story problems related to integers is that these tend to be quite accessible without the use of negative numbers. In fact, problems in real-world contexts can be solved without invoking negatives, and this generally seems to be more natural. For example, in the Debate Club problem above, the use of integers is entirely unnecessary. The debate teamed earned $300 from a car wash, and they spent $25 on supplies. So, their profit is $300 - $25 = $275. The only reason to represent this situation as involving addition of a negative, rather than whole-number subtraction, is the book's instruction to do so. (In fact, the same story problem would be identified as a subtraction problem if it appeared in a second- or third-grade textbook.) Furthermore, even when the problem is framed as involving the addition of two integers, the way that this "addition" is performed is to subtract!

The character of the role of the context here is not that integers are useful for solving problems in certain contexts. Rather, the message we get is that there are contexts than *can* be described in terms of integers if we choose to do so. In other words, integers do not appear as tools that are necessary for solving problems in the world, but as an artificial, purely mathematical lens that we can apply to real-world contexts when that is the game to be played. We note that this is not merely a matter of poorly chosen story problems. Two years into a project concerning integers, our team has yet to identify a real-world context that actually requires one to use negative numbers. When one considers that Greek mathematicians, for

example, solved a wide variety of real-world problems without the use of negative numbers, the lack of appropriate contexts for which negative numbers are necessary seems less surprising.

In most textbooks, students are expected to translate a story problem into an expression involving integers (as in the example above) and then to evaluate that expression by appealing to sign rules or to a procedure associated with a prescribed model. That is, we do not see students' intuitions about the contexts being tapped to guide solution strategies (beyond the task of translating to a number sentence). Furthermore, it is difficult to find power in the rules that students are expected to use to evaluate integer expressions. Although algebraic equations may enable students to solve problems that they could not solve otherwise, number sentences involving integers represent extra work required of students in the name of solving otherwise readily accessible tasks.

We do not mean to suggest that the only reason to relate integers to contexts would be that they provide indispensable tools for solving problems. There could be legitimate pedagogical purposes for doing so. Certainly, experts in reasoning about integers are capable of relating them to real-world contexts. However, in our review of textbook treatments of integer topics, we have found that these contexts tend to be used in a contrived fashion that may undermine the utility of the integers *within* mathematics.

Children's Intuitions About Integers

We look to interviews with elementary children to reconsider opportunities for learning about negative numbers. We have seen powerful examples of children's reasoning about positive and negative numbers prior to formal instruction about integers. We discuss the reasoning of two children, a first grader (Jackson) and a fourth grader (Roland), in the context of solving open number sentences. (We use pseudonyms for the children's names.)

Jackson's Reasoning

Jackson, a first grader, had heard of negative numbers and had seen a number line that included some negatives in his classroom. He had received no formal instruction concerning negative numbers. This minimal exposure to the notion of negatives was enough to enable him to extend his counting strategies to the left of zero. For example, Jackson solved -2 + 5 = ☐ by starting at -2 and counting up five: -1, 0, 1, 2, 3. Although he had not been taught to solve addition problems involving negatives, Jackson's previous experiences with addition enabled him to solve this novel problem.

Early in his interview, Jackson had said that he could not solve problems such as 6 + -3 = ☐. He had no way of making sense of starting with six of something and then adding negative three. However, following Jackson's solution above (-2 + 5 = 3), he was presented with 5 + -2 = ☐. On the basis of his answer that -2 + 5 = 3, Jackson concluded that 5 + -2 also equals 3, explaining (in the language of a first grader) that the results must be the same because of the commutative property of addition. When asked what 5 + -2 meant, Jackson decided that it meant the same as subtraction since 5 – 2 was also 3. He went on to solve the problem 6 + ☐ = 4 by thinking about how much he would have to subtract from 6 to get 4.

Roland's Reasoning

Roland was in fourth grade. Like Jackson, he had heard of negative numbers but had not received formal instruction concerning operations involving them. Nonetheless, Roland reinvented common sign rules for addition and subtraction of integers by reasoning about

negative numbers as "the opposite" of positives. For example, Roland solved -5 + -1 = ☐ by reasoning that adding two negative numbers "would make it further from the positive numbers." Since he knew that the sum of 5 and 1 was 6, he reasoned that -5 plus -1 should equal -6. Roland was also able to solve -5 – -3 = ☐ by building on his previous reasoning. He said, -5 – -3 "would probably be… minus two, because if you use addition… it would be farther from positive numbers, so if you do the opposite it should be closer." Roland reasoned that adding -3 to -5 would result in moving further into the negatives, so subtracting -3 from -5 should result in moving toward the positives instead. The opposite operation had to have the opposite effect.

Mathematical insights. Jackson had productive insights concerning the properties of integers. He arrived at these by drawing on basic understandings of the natural number domain, together with simple familiarity with the notion of negative numbers as numbers less than, and to the left of, zero. He made progress by reasoning within the realm of mathematics, not by looking outside of it. Similarly, Roland used logic, together with an understanding of + and – as opposites, to effectively reinvent sign rules for addition and subtraction of negative numbers. Middle school textbooks go to elaborate lengths to present these rules in such ways that hopefully students will remember them. Roland, a fourth grader, figured them out for himself.

Roland and Jackson did not make the leaps described above by relating integers to a context or by applying a chip model. Rather, they leveraged logic and familiar mathematical properties to solve novel problems. These moves opened new possibilities for them. At an elementary level, these children's productive insights echo the work of mathematicians such as Newton, who overcame difficulties associated with negatives through a pragmatic approach, driven by mathematical utility and logical necessity.

Implications

In light of the history of the integers, it seems worthwhile to reconsider the ways that contexts and models are typically used in integer instruction. In the history of mathematics, attempts to relate integers to contexts and models both facilitated and constrained progress. The notion of negative numbers arose not out of real-world situations involving directed magnitudes, but from within mathematics itself. Ultimately, mathematicians made sense of and accepted negatives as numbers not because of any relationship to real-world quantities, but on purely mathematical terms. Resistance to negatives was alleviated by a shift in perspective. Mathematicians let go the search for clarifying models and came to view numbers as abstract entities. They recognized that the natural numbers and the integers are distinct domains with different properties and that the integers are useful for solving mathematical problems (as opposed to story problems).

Our purpose is not to denigrate contexts, models, manipulative materials, or textbooks. On the contrary, when it comes to many topics, we support approaches that relate numbers to quantities. In the case of young children learning to reason in the natural number domain, we see real-world contexts and materials such as Unifix cubes and base-ten blocks as extremely helpful resources. However, integers are different! The notion of a negative number is a strange idea and one that does not correspond well with the experiences of people in the world. Even individuals who have come to accept negative numbers within mathematics tend to regard them as fictitious entities with respect to the physical world. One child that we interviewed described negatives as belonging to a "ghost world" in which faint images of objects could be seen, although the objects were not really there.

It is no wonder that attempts to relate integers to contexts often appear contrived. Why do we insist on such an approach? Consider instead an honest, purely mathematical alternative whereby students are asked to confront issues that were previously swept under the rug: *Why can't we subtract bigger from smaller? What if we could? We can add and multiply without restriction. Suppose that we wanted to be able to subtract and divide with as little restriction as necessary. How could we do that, and how might we interpret the results?* These are fair, mathematical questions that children are capable of exploring. Rather than think up elaborate ways of getting students to learn and remember the sign rules for operations with integers, we could invite children to establish for themselves what those rules ought to be. Preliminary evidence from interviews indicates to us that children could come to construct the integers from the natural numbers, given appropriate support for doing so.

Endnote

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