An Analysis of Mathematical Content Knowledge for Teaching

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Two Tasks

Below is the work of Terry, a second grader, who solved this addition problem and this subtraction problem in May.

**Problem A**

\[
\begin{array}{c}
1 \\
259 \\
+ 38 \\
\hline \\
297
\end{array}
\]

**Problem B**

\[
\begin{array}{c}
31 \\
429 \\
- 34 \\
\hline \\
395
\end{array}
\]

In March, Andrew, a second grader, solved 63 - 25 = \[ \square \] as shown below.

\[
\begin{array}{c}
- 63 \\
- 25 \\
\hline \\
- 20 \\
\hline \\
38
\end{array}
\]

- Does the 1 in each of these problems represent the same amount? Please explain your answer.
- Explain why in addition (as in Problem A) the 1 is added to the 5, but in subtraction (as in problem B) 10 is added to the 2.

- Explain why Andrew's strategy makes mathematical sense.
- Please solve 432 - 162 = \[ \square \] by applying Andrew's reasoning.
Activity

• Take a few minutes to think about the two tasks.

• Think about how you solved each task. What knowledge did you draw upon in solving them?

• If time remains, talk to a neighbor.
Two Tasks

Below is the work of Terry, a second grader, who solved this addition problem and this subtraction problem in May.

**Problem A**

\[
\begin{align*}
\phantom{1} & \quad 2 \quad 5 \quad 9 \\
+ & \quad 3 \quad 8 \\
\hline
& \quad 2 \quad 9 \quad 7
\end{align*}
\]

**Problem B**

\[
\begin{align*}
3 & \quad 1 \\
\phantom{1} & \quad 4 \quad 2 \quad 9 \\
\hline
& \quad 3 \quad 4 \\
- & \quad 3 \quad 4 \quad 5 \\
\hline
& \quad 3 \quad 8
\end{align*}
\]

In March, Andrew, a second grader, solved 63 - 25 = □ as shown below.

\[
\begin{align*}
6 & \quad 3 \\
\phantom{1} & \quad \underline{2} \quad 5 \\
\hline
& \quad \underline{3} \quad 8
\end{align*}
\]

- Does the 1 in each of these problems represent the same amount? Please explain your answer.
- Explain why in addition (as in Problem A) the 1 is added to the 5, but in subtraction (as in problem B) 10 is added to the 2.
- Explain why Andrew’s strategy makes mathematical sense.
- Please solve 432 - 162 = □ by applying Andrew’s reasoning.
Types of Knowledge

• Common content knowledge (CCK)
  • The knowledge commonly taught in school and used in life.

• Specialized content knowledge (SCK)
  • The knowledge that teachers need, not because they teach it to students but because it serves as the foundation for common content knowledge.

• Pedagogical content knowledge (PCK)
  • The knowledge at the intersection of the content and students’ thinking.
Types of Knowledge: Examples

• **Common content knowledge (CCK)**
  • *Evaluate and understand the meaning of $12 \div 3$.*

• **Specialized content knowledge (SCK)**
  • *Write a real-life story problem that could be represented by the expression $12 \div 3$.*

• **Pedagogical content knowledge (PCK)**
  • *How might children think about the problem you wrote?*
Principal Investigators  Randy Philipp, PI
                    Vicki Jacobs, co-PI
Faculty Associates          Lisa Clement Lamb
                          Jessica Pierson Bishop
Graduate Student          John (Zig) Siegfried
Research Associate        Bonnie Schappelle
Project Coordinator       Candace Cabral

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The opinions expressed in this presentation do not necessarily reflect the position, policy, or endorsement of the supporting agency.
# STEP Participant Groups

(30+ per group)

<table>
<thead>
<tr>
<th>Category</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Prospective Teachers</strong></td>
<td>Undergraduates enrolled in a first mathematics-for-teachers content course</td>
</tr>
<tr>
<td><strong>Initial Participants</strong></td>
<td>0 years of sustained professional development</td>
</tr>
<tr>
<td><strong>Advancing Participants</strong></td>
<td>2 years of sustained professional development</td>
</tr>
<tr>
<td><strong>Emerging Teacher Leaders</strong></td>
<td>At least 4 years of sustained professional development and some minimal leadership activities</td>
</tr>
<tr>
<td><strong>Strong Mathematics Students</strong></td>
<td>STEM students, with no teaching intentions, enrolled in upper-division mathematics courses</td>
</tr>
</tbody>
</table>

**K–3 Teachers**
### Mathematical Understanding in the Andrew Task

<table>
<thead>
<tr>
<th>Score</th>
<th>Evidence shown</th>
</tr>
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<tbody>
<tr>
<td>0</td>
<td>No evidence of mathematical understanding</td>
</tr>
<tr>
<td>1</td>
<td>Primarily procedural knowledge; none-to-minimal evidence of conceptual understanding</td>
</tr>
<tr>
<td>2</td>
<td>Procedural knowledge and conceptual understanding evident</td>
</tr>
<tr>
<td>3</td>
<td>Strong procedural knowledge and conceptual understanding evident; justification incomplete</td>
</tr>
<tr>
<td>4</td>
<td>Strong and related procedural knowledge and conceptual understanding evident; justification complete</td>
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Andrew and Ones Tasks

Which score for which group?

<table>
<thead>
<tr>
<th>Group</th>
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<tr>
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<td>?</td>
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<tr>
<td>SMS</td>
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Each response was scored on a 5-point scale (0–4).

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Each response was scored on a 5-point scale (0–4).
Findings

• We posed eight tasks to the participants.
• On six tasks, the SMSs performed as well as the ETLs.
Please provide solution strategies—as many as you can—that you might expect children to use to solve the following problem:

*Pablo read 15 pages of his library book on Saturday. The book has 32 pages. How many pages will he have to read on Sunday to finish his book?*
Ones Task

Whereas SMSs cannot answer most tasks, such as the Andrew Task, without grappling with the meaning of the dash 2, they could answer the Ones Task in a purely calculational way without grappling with the place-value ideas embedded in the task. So, for example, they see the 1 in Problem A as a 1 added to the 5 and 3, and they see the 1 in Problem B as a 10 that creates 12 from which 3 is subtracted. By the way, this task presents implications for those of us teaching college students: To have students grapple with underlying concepts, at least initially, constrain the tasks so that they cannot be answered without unpacking the meanings.
Findings

• The SMSs scored as well as the ETLs on the other six tasks.
• Does that show that they hold the same Mathematical Content Knowledge?
The Land of SCK

How do people reach the land of specialized content knowledge?
Does the path one takes to get to SCK have implications for teaching?
The Land of SCK

A Mathematical Road to SCK

A Road to SCK Through Children’s Mathematical Thinking
Project Z: Mapping Developmental Trajectories of Students’ Conceptions of Integers, 2009–present

Principal Investigators
Lisa Lamb, Jessica Bishop, & Randolph Philipp

Faculty Associate
Ian Whitaker

Graduate Students
Spencer Bagley & Casey Hawthorne

Project researchers
Bonnie Schappelle, Mindy Lewis, & Candace Cabral

Undergraduate Students
Kelly Humphrey, Jenn Cumiskey, Danielle Kessler

Funded by the National Science Foundation,
DRL-0918780
Ways of Reasoning (WoR)

Solve the following and remember how you thought:

• $-2 + □ = 4$
• $-5 + -1 = □$
• $5 - □ = 8$
Teachers’ Approaches to Integer Tasks

Teachers possess these rich ways of reasoning (WoR) about integers. They are

• Able to flexibly apply WoR and
• Able to analyze problems and strategically combine WoR.
Kalani: \[ -3 - \square = 2 \]

*First, Kalani thinks of moving from -3 to 2, as if the number sentence were \(-3 + x = 2\).*

“So I am thinking about the number line .... So I am starting somewhere ... and what do I do to end up at positive 2? I am moving 1, 2, 3, 4, 5—five units to the right.”

*Second, Kalani accounts for the subtraction sign.*

“I am moving the opposite direction [because of the subtraction sign], so I would write down negative 5 here.”
Pedagogical Goals

Procedure Only

• Stated rules clearly
• Expected students to practice rules and demonstrate their use

“They would all do that or they would get it wrong. I call my classroom “Smithsville,” and you do it the Smiths way.”
Pedagogical Goals

Procedure Only
- Stated rules clearly
- Expected students to practice rules and demonstrate their use (Smithsville)

Procedural With Conceptual
- Professed valuing *conceptual understanding* but were unable to explicate what that might entail.
  “I want them to reason through it as much as they can. They need to know that there is some other reason [beyond the rule].”
  “... just kind of picture it in their head.”
  “... have a good understanding of what positives and negatives mean.”
Pedagogical Goals

No relationship was found among teachers between their
a) flexibility /robustness of reasoning and
b) stated pedagogical goals.
Pedagogical Goals

• Did we expect teachers to mention ways of reasoning as goals for instruction?
  • No!
• Did we expect teachers to apply ways of reasoning?
  • We did not know what to expect.
• Did they apply ways of reasoning?
  • Yes!
Interpreting Student Thinking
Grade 4  \(-5 + -1 = \square\)

• “Minus 5 plus minus 1 equals minus something ...; 5 plus 1 equals 6, so this is minus 6 .... If you add these two together, it makes it farther from the positive numbers.”
Interpreting Student Thinking
Grade 4  -5 + -3 =☐

• “Minus 5 minus minus 3 equals something. It would probably be something. It would probably be minus 2 because if you add ... . If you use addition with this, it would be farther from the positive numbers, so if you do the opposite, it should be closer.”
Interpreting Student Thinking

1/3 Difficulty understanding student reasoning

Kalani: “I think he got confused. There’s no context involved...So I don’t see a clear understanding on his part at all.” (referring to \(-5 + -1 = \square\))

“I don’t quite understand him when he used the opposite. Opposite of what? ...There is no context.” (referring to \(-5 - -3 = \square\))
Interpreting Student Thinking

Appreciating Understanding

1/3 Difficulty understanding student reasoning

1/3 Struggled appreciating nuances, either overgeneralizing or being overly critical.

“He understands the rule that adding a negative and subtraction are exactly the same.”
Interpreting Student Thinking

Correlation?

Teachers’ Ways of Reasoning
- No

Pedagogical Goals
- Yes

Interpreting Student Thinking

Appreciating Understanding
Professional Noticing of Students’ Mathematical Thinking

- Attending
- Interpreting
- Deciding how to respond
Specialized Content Knowledge

• What is specialized content knowledge (SCK)?
  SCK is mathematical knowledge that teachers hold that they do not explicitly teach but that enables them to effectively teach their students.

• SMSs’ knowledge of Andrew Task
• 7th-grade teachers’ knowledge of integer reasoning
Discussion

A Mathematical Road to SCK

A Road to SCK Through Children’s Mathematical Thinking
Brian solves \(-9 + x = -4\)

“Because I saw the negative 4 and I know negative 9 is smaller than negative 4, so I knew it had to be positive. ... So how do you go from 9 to 4? You have to subtract 5.”

Although to us Brian seems to be using reasoning, he repeatedly asserts that he is using underlying rules.