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Coordinating multiple representations of polynomials: What do patterns in students' solution strategies reveal?

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ABSTRACT

We investigate the strategies used by 64 advanced secondary mathematics students to identify whether a given pair of polynomial representations (graphs, tables, or equations) corresponded to the same function on an assessment of coordinating representations. Participants also completed assessments of domain-related knowledge and background skills. Cluster analysis of strategies by representation pair revealed patterns in the participants' strategy use. Two clusters were identifiable on tasks that required matching equations to graphs and graphs to tables. We identified overlap between these two clusters, suggesting that while the representation pair influenced strategy choice, there was also a general distinction between students who used more and less sophisticated strategies. However, students who used more sophisticated coordination strategies were similar to the others on measures of domain-specific knowledge or background skills. We consider implications for future investigations testing interventions to promote coordinating representations.

Problems in advanced secondary mathematics often require students to coordinate multiple external representations of functional relationships (Chang, Cromley, & Tran, 2016; Ferrara, Pratt, & Robutti, 2006; Janvier, Girardon, & Morand, 1993; Leinhardt, Zaslavsky, & Stein, 1990; Moschkovich, Schoenfeld, & Arcavi, 1993). Teaching with multiple external representations can foster student understanding of important mathematical concepts and relationships (Brenner et al., 1997), and school mathematics policy documents recommend teaching with multiple representations (Department for Education, 2013; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). The recommended focus on multiple representations builds upon decades of research by educational psychologists (Ainsworth, 2006; Mayer & Moreno, 2002; Rau, 2016) and mathematics educators (Acevedo Nistal, Dooren, Clarebout, Elen, & Verschaffel, 2009; Brenner et al., 1997; Hiebert & Carpenter, 1992; Leinhardt et al., 1990; Parnafes & Disessa, 2004; Yerushalmy, 1991) showing the benefit of multi-representational approaches. Accordingly, numerous teachers (e.g., Eichler & Erens, 2014) and curriculum authors (Chang et al., 2016) target skills such as constructing, coordinating, and reasoning with multiple representations as goals for instruction.

In order to design effective interventions to promote students' skills in coordinating multiple representations (CMR), researchers must identify related knowledge bases (Rau, 2016) and must also identify effective coordination strategies (Ainsworth, 2006). This study uses cluster analysis (Milligan & Hirtle, 2003) to identify profiles of CMR strategy use, and it extends prior research which has shown that students must have some domain-specific knowledge to coordinate representations in technical disciplines (Rau, 2016). We focused this initial work on representations of functions common in secondary mathematics: linear, quadratic, and cubic polynomials.

CMR strategies are actions used to identify whether two representations correspond to the same underlying function. For example, Fig. 1 shows a sample CMR task with an equation and graph. One CMR strategy is matching points on the graph with coordinates generated from the equation. Alternatively, a student might identify that the shape of the graph does not match the degree of the equation. Identifying and coding strategies is one way

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Do the equation and graph represent the same function?

$f(x) = -2x^2 + 25$

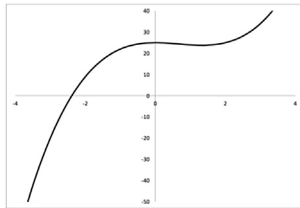

Fig. 1. Sample equation-graph item.

to begin understanding CMR skills. Analysis of strategy profiles helps identify patterns in strategy use. In what follows we present the results of a cluster analysis of the strategies used by 64 advanced secondary mathematics students to coordinate pairs of polynomial representations presented as items similar to Fig. 1. The results of this analysis suggest directions for future interventions designed to develop coordination skills.

1. Conceptual framework and prior research

Ainsworth's (2006) framework on multiple representations undergirds our focus on CMR, and Siegler's work on strategy selection (Siegler, 2005) motivated our decision to examine how students use strategies while solving the CMR tasks.

1.1. CMR and coordination strategies

Ainsworth's (2006) DeFT (Designs, Functions, Tasks) framework integrates research on teaching and learning with multiple external representations. Ainsworth, citing Yerushalmy (1991), described how teaching students to coordinate representations of functions in school mathematics is non-trivial. Rau's (2016) review suggests that efforts to teach CMR should account for both knowledge of individual students and socio-cultural characteristics of representation usage. Acevedo Nistal et al. (2009) argue that problem solving strategies and representational flexibility are connected to both the characteristics of the representations in use and the characteristics of the students interacting with the representations. This study builds on these frameworks through focusing on CMR task demands and knowledge. Less work in the area of CMR has explored how students deploy strategies to coordinate representations, and how these strategies are connected to characteristics of the representations that are considered. In order to explore profiles of strategy use in relation to the characteristics of representations we drew on Siegler's work on strategy selection.

Siegler's overlapping waves theory (2005) describes how, in general, learners use more sophisticated strategies across development. However, learners who can use a more sophisticated strategy may use a less sophisticated strategy on some problems, meaning strategy choice is not determined by level of development. For example, on the problem $3 + 8$, a child who has used the more sophisticated strategy of counting on from the larger summand may continue counting on from the smaller summand in some subsequent trials. Siegler notes that students choose strategies that “fit the demands of problems and circumstances and that yield desirable combinations of speed and accuracy, given the strategies and available knowledge that children possess” (Siegler, 2005, p. 771). Recent research has used Siegler's approach to analyze problem solving strategy choice among elementary and secondary mathematics students (Booth, Lange, Koedinger, & Newton, 2013; Jurdak & El Mouhayar, 2014), and we extend that work here in the area of CMR.

In contrast with Siegler's approach which studied the development of strategies to solve one type of problem, we use cluster analysis to *explore* profiles of strategy use for solving *different* types of problems. This exploratory work requires approaches like cluster analysis rather than variable-centered approaches like regression. This analytical technique, which is relatively rare in studies of strategy analysis, is described in more detail in the Methods section.

1.2. CMR strategies and views of function

Three of the most common function representations in school mathematics are graphs, tables, and equations. A review of literature on CMR skills identified that many secondary mathematics students struggle to coordinate representations with graphs (Chang et al., 2016; De Bock, van Dooren, & Verschaffel, 2015). As students learn to use and interpret graphs, one important development is transitioning from making point-by-point comparisons to more holistic comparisons of functions and graphs (Friel, Curcio, & Bright, 2001; Leinhardt et al., 1990; Yerushalmy, 1991). Making point-wise connections reflects a “process” view of a function, while global comparisons treat functions as “objects” (Moschkovich et al., 1993). Given nearly any representation pair, it is possible to identify whether two representations are the same function by matching ordered pairs (using a process view), but other strategies may yield accurate answers in less time. For example, as in Fig. 1 above, a student may answer more accurately and quickly by evaluating the global features of the function. Global features include the slope (for linear functions), direction, or degree (using an object view). Friel et al. (2001) suggest that comparing equations and graphs using a point-by-point method is a less sophisticated strategy than using global properties. However, this distinction may not necessarily apply to the coordination of tables and equations, where point-by-point comparisons are the only feasible option. In cases where point-by-point matching is the only option, however, point-wise CMR strategies may still vary by expertise. This highlights the importance of considering the representation pair in analyses of CMR strategy choice.

CMR strategies influence both problem solving speed and accuracy. That there is a tradeoff of speed and accuracy in problem solving is well documented (Wickelgren, 1977). Increasing speed generally decreases accuracy, while focusing on increased problem solving accuracy can slow performance. However, the speed-accuracy relationship is moderated by the strategy used to solve a problem. Some strategies allow problem solvers to increase accuracy while simultaneously maintaining or increasing increased speed. In this study, global comparisons of a function's shape or direction allow for relatively fast and accurate CMR. In contrast, point-by-point comparisons can be accurate, but time consuming, particularly when many values are calculated.

1.3. Summary

Prior research indicates that CMR strategy choice is related to both the level of student development in the domain and student characteristics such as background knowledge and skills. In the case of CMR, this study builds on prior work that has investigated connections between students' background skills, their domain-specific knowledge and their success coordinating multiple representations (Ainsworth, 2006; Cromley et al., 2017; Rau, 2016). We address the following research questions:

1. What strategies do secondary students use to solve polynomial CMR tasks?
2. How are students' strategy profiles different by representation pairs in CMR tasks?
3. Is it possible to identify distinct groups of students based on the proportion of strategies used? If yes, then
   a. Do the groups fall along a continuum from less sophisticated to more sophisticated strategies for CMR?
   b. Is one group more successful than the other on background characteristics and related measures of mathematical proficiency?

2. Methods

2.1. Participants

Participants were 64 advanced secondary mathematics students enrolled in calculus and pre-calculus attending suburban high schools in the Mid-Atlantic US. The mean age of the participants was 16.9 ($SD = 0.8$) at the time of participation. The sample consisted of 60% female and 79% white students. As a proxy for socioeconomic status, we measured maximum parental education; 80% of families had at least one parent with a Bachelor's degree or above.

2.2. Measures

Participants completed multiple measures, including a demographic questionnaire, a series of paper-and-pencil measures on spatial skills, a researcher-designed test of calculus conceptual knowledge, measures of graph and table skills and calculus proficiency, and, finally, a CMR measure in which participants verbalized their thoughts while attempting to coordinate two polynomial representations. Detailed descriptions follow.

2.2.1. Spatial skills measures

Participants were presented with measures of spatial skills because these malleable skills may be related to CMR success (Cromley et al., 2017; Höffler, 2010; Uttal et al., 2013; Wai, Lubinski, Benbow, & Steiger, 2010). Three spatial skills measures, a Mental Rotations Test (MRT), a Paper Folding Test (PFT), and a Hidden Figures Test (HFT) were administered to participants. We used the *MRT-A* in a CAD-redrawn version of Vandenberg and Kuse (1978) from Peters et al. (1995). Participants were instructed to find the two rotated versions of a "target" 3-D figure from four figures. Following the guidelines for administering the MRT-A, participants were given three minutes to complete all 12 items. The *MRT-A* has been associated with diagram comprehension in prior research with 157 undergraduates (Voyer & Hou, 2006), and shows good reliability (Cronbach's $\alpha = 0.910$). We found evidence for good internal consistency with our sample (Cronbach's $\alpha = 0.831$).

We provided three minutes for the participants to complete the first 10 items of the *PFT* (Ekstrom, Frech, Harman, & Derman, 1976), a measure of spatial visualization from Educational Testing Services. In each item participants were shown five diagrams of a sheet of paper with one to three folds, and another diagram of a square sheet with a hole punched in it. Participants were asked to identify which one of the five unfolded sheets would match the diagram of the hole-punched folded sheet. Ekstrom, French, and Harman (1979) found that the *PFT* had good reliability with high-school aged samples (Cronbach's $\alpha = 0.84$ with $N > 2500$). We also found evidence for good internal consistency in the present study (Cronbach's $\alpha = 0.700$).

We provided three minutes for the participants to complete the first 16 items of the *HFT* (Ekstrom et al., 1976), another measure of spatial visualization by the Educational Testing Service. For each item, participants were asked to identify a simple two-dimensional figure embedded within a complex geometrical figure. The *HFT* has been found to have good reliability with 40 college-aged students (Cronbach's $\alpha = 0.88$; Stankov, 1988). We did not find acceptable internal consistency with our sample (Cronbach's $\alpha = 0.325$), and therefore excluded the *HFT* scores from further analyses.

2.2.2. Calculus conceptual knowledge measure (CCM)

We constructed a 32-item measure assessing students' conceptual understanding of calculus, including derivatives, functions and limits, and the chain rule. Many of these items were adapted from released Advanced Placement® Calculus AB exams.[1] Each of these multiple-choice items emphasized understanding conceptual relationships rather than calculations. We found evidence for excellent internal consistency reliability with our sample (Cronbach's $\alpha = 0.937$).

2.2.3. Graph/table skills measure (GTS)

We constructed a measure comprised of six released graph items and five released table items from National Assessment of Educational Progress (Grade 12), National Assessment of Adult Literacy, and Adult Literacy and Lifeskills survey (National Center for Education Statistics, 2017). The selected multiple-choice items were from the "Easy" groups. Participants were given six minutes to complete the measure. We did not find evidence for acceptable internal consistency with our sample (Cronbach's $\alpha = 0.347$), and therefore excluded the Graph/Table Skill measure from further analyses.

2.2.4. Pre-calculus conceptual measure (PCA)

To measure students' conceptual understanding of multiple representations of functions, we used the eight-item "Understand function representations" subscale of the Pre-Calculus Concept Assessment (Carlson, Oehrtman, & Engelke, 2010) purchased from Rational Reasoning, Inc. The paper-and-pencil measure has been well validated with students in middle school through college; prior research provided extensive evidence supporting validity and reliability (Cronbach's $\alpha = 0.730$). We found evidence for acceptable internal consistency in the present study (Cronbach's $\alpha = 0.692$).

2.2.5. Calculus proficiency measures

To measure calculus proficiency, we compiled a measure with 11 released Advanced Placement® Calculus AB exam multiple-choice questions. In contrast with the conceptual measure, this assessment included a mixture of calculations and manipulations. Participants were given 15 min to complete the measure. We found evidence for good internal consistency with our sample (Cronbach's $\alpha = 0.716$).

2.2.6. CMR measure and concurrent think-aloud protocol

To assess students' coordinating multiple representation skills and investigate what strategies students used for CMR tasks, we administered 12 coordinating multiple representations (CMR) items on a Tobii T60 eye-tracking apparatus.[2] We obtained data on time used to complete the CMR items from the eye-tracker device. Participants concurrently followed a think-aloud protocol (Ericsson & Simon, 1998) while completing the CMR measure. Participants were asked to read a pair of representations (e.g., a graph and an

[1] Advanced Placement® is a trademark registered and/or owned by the College Board, which was not involved in the production of, and does not endorse, this research.

[2] In this paper, eye-tracking data are not analyzed.

equation) shown on the screen of the eye-tracker and to determine whether or not each pair of representations expressed the same underlying function. Fig. 1 shows one of the items used in the assessment. As presented in Table 1, four questions presented equation-graph pairs, four questions presented equation-table pairs, and the remaining four questions were graph-table pairs. We do not distinguish the order of presentation because both representations were visible simultaneously. Six CMR items matched, and six did not. The matched and mismatched items were counter-balanced within representation pair types. We found acceptable internal consistency reliability for this 12-item scale with our sample (Cronbach's $\alpha = 0.650$).

2.3. Procedure

We obtained parent consent and student assent, then administered the study measures to participants individually in a session for approximately 70 min at school during non-class time (e.g., extracurricular activity sessions). Students received gift cards in the amount of $10 as compensation for their participation in the study. After completing all paper-and-pencil measures, the participants completed the CMR measure while following a think-aloud protocol. While participants read the prompts, the equations, and the reminders, we reminded the participants to read out loud. Most importantly, as the participants solved the problems they were reminded to verbalize their thoughts.

2.4. Data and coding

2.4.1. Transcription and coding of think-alouds

The think-aloud protocols were transcribed verbatim. The transcripts were coded for strategy use and verbalized arithmetic. The codes were applied to each utterance, roughly corresponding to a phrase containing a subject and a verb such as “the negative x squared matches the parabola.” Each utterance received one and only one code. One coder coded all utterances ($N = 2344$ coded utterances), and a second coder re-coded the transcripts of 35% of participants. The second coder was trained on the 65% of the transcripts that were not double-coded until 100% agreement was reached. For the double-recoded transcripts ($N = 1166$ coded utterances), the two coders agreed on 995 utterances (85.3%), yielding a Cohen's $\kappa = 0.827$.

2.4.2. Coding for strategy use

Research question 1 is what strategies do secondary students use to solve polynomial CMR tasks? We answered this question through coding the think aloud protocols. The codes for strategy use were developed using a combination of a priori and emergent codes. A priori codes focused on the two major strategies available: local point-by-point comparisons and global evaluations (Friel et al., 2001; Moschkovich et al., 1993). These initial codes were refined in order to capture the various strategies used. Ultimately, we coded seven main strategies used by students while solving the CMR questions.

Matching ordered pairs (MOP), was coded when a participant coordinated an ordered pair across the two representations (i.e., “point testing”). For a graph-table problem, a participant who matched ordered pairs would verbally compare each ordered pair in the table with its equivalent point on the graph. For an equation-graph or table-graph pair, MOP was coded when the participant verbalized the y-value for a given x-value and located the corresponding ordered pair on the graph. Many times when students used the MOP strategy, they verbalized arithmetic calculations such as “negative 2 times 1 squared plus 25 is 27.” Such verbalized arithmetic was coded separately (see below).

Matching intercepts is a common strategy secondary students use to solve function coordination problems (e.g., Moschkovich, 1999). Therefore, we created distinct codes for instances of matching intercepts. When MOP strategies involved the x-intercept or y-intercept *without the participant mentioning the word intercept*, these utterances were coded as MOPX and MOPY, respectively. When students explicitly named the x-intercept or y-intercept when matching ordered pairs, these utterances were coded as MINTX or MINTY. In the case of MOPX or MOPY it was not clear whether the student was attending the intercept, or if the intercept happened to be next in the sequence of points the student was matching. In contrast, MINTX and MINTY indicate the student explicitly attended to the intercept. Instances of MOP, MOPX, MOPY, MINTX, and MINTY were mutually exclusive, and these five strategy codes were considered as individual variables in further analyses.

The sixth strategy code describes a participant evaluating (EVAL) global properties of the function like the order (degree), the direction, the magnitude of the leading coefficients, or the magnitude of the constants. Table 2 shows examples of various utterances coded as EVAL. Initially we coded EVAL subcodes for specific evaluation strategies (e.g. comparing the degree of an equation to the shape of a graph). However, not all representation pairs allowed for all sub-codes of EVAL, so for the purpose of this analysis we combined all EVAL codes because they all indicate attending to global properties of the graph.

Finally, the last set of codes involves participants checking their work (CHECK), which were mainly verbalizations of verifying a

Table 1
Summary of items on CMR assessment.

Item	Representation pair[a]	Polynomial degree[b]	Direction match?	Y-intercept match?	Did functions match?
1	E-G	2,3	N	Y	N
2	G-E	1,1	Y	Y	Y
3	T-G	3,3	Y	Y	Y
4	E-T	3,3	Y	Y	N
5	G-T	1,1	Y	Y	N
6	T-E	2,2	Y	Y	Y
7	E-G	3,3	Y	Y	Y
8	E-T	2,2	Y	Y	Y
9	T-E	3,3	Y	Y	N
10	G-E	2,2	N	Y	N
11	G-T	1,1	Y	Y	Y
12	T-G	3,3	Y	Y	N

[a] The pair of representations shows both the representation type (E = Equation, G = Graph, T = Table) as well as the left-right order of presentation. In the analysis we treat E-G and G-E as the same pair of representations.
[b] The degree of the polynomials shows the degree from left to right. For example, Question 1 showed an Equation and Graph where the equation was degree-2 and the graph corresponded with a degree-3 function.

Table 2
Definitions of and examples of coded utterances from think-aloud data.

Code	Definition	Example
MOP	Matching ordered pairs, not including those that are intercepts	"Let's see, it's 2, put it on the graph, equals negative 20."
MOPX/MOPY	Matching an ordered pair that is the x-intercept (MOPX), or y-intercept (MOPY) but without the student mentioning the word intercept	"Ah … Looking at the graph, looks like when *x* equals 3, *y* equals 0. So when I plug 3 into the equation. 3 times 2 equals 6.6 plus 6 is 12." "So when *x* is equal to zero the function for *y* should be negative 20, which it is."
MINTX/MINTY	Matching x-intercept (MINTX) or y-intercept (MINTY). When students explicitly refer to the intercept	"We set the function equal to zero, that is 2*x* equals 6 so *x* equals 3 and that is in fact the *x*-intercept." "It says plus 20 when it looks to be around 25 for the *y*-intercept."
EVAL	Evaluating direction/order/magnitude	"Values are increasing at a rate of 2 in the table and it seems to be in the graph." "To me that doesn't look like an *x* squared function that looks to be like *x* cubed." "It looks like the slope should be steeper."
CHECK	Re-evaluating a statement or decision previously made or verbalizes the need for more information	"I'm going to look at *x* is negative 3 and find that on my graph and see where ah, yea it looks like that's about the right *y* value for the negative 3, ah maybe slightly off, it looks like it's probably more like 5."

calculation or verbalizations of "let me check more" points.

2.4.3. Coding for arithmetic

Participants frequently verbalized calculations. As an additional measure of mathematical proficiency, we coded students' utterances of correct arithmetic calculations as Arithmetic-Correct (ARIC). Incorrect arithmetic calculations were coded with Arithmetic-Incorrect (ARII).[3] We examined the percentage of correct arithmetic calculations over all arithmetic calculations articulated by participants within each CMR question type (e.g., % of correct arithmetic calculations = ARIC/[ARIC + ARII] for equation-table questions).

2.4.4. Time used for CMR items

Finally, we used time stamped data to calculate the mean time from presentation of a problem until the student stated an answer to each CMR item.

2.5. Data analysis

Table 3 shows the descriptive statistics of the coded think aloud data including time used, frequency of verbalizations of strategy use, and accuracy answering each type of CMR question separately across question types—equation-graph, equation-table, and graph-table. Table 3 also contains descriptive statistics for the paper and pencil measures. One notable observation in Table 3 is that students used approximately the same number of strategies for the equation-graph and equation-table items, while they used more strategies per graph-table item. We examined the partial correlations between the frequency of verbalizations of strategy use and accuracy, controlling for time used, to understand whether the frequency of verbalizations of strategy use had an association with accuracy of answers for equation-graph, equation-table, and graph-table questions (Table 4).

We also explored correlations among the percent of strategies used. Two strategies, MOP and EVAL had a negative correlation (r [65] = -0.30, p = 0.030). One plausible explanation for this negative correlation is that students who use MOP are less likely to use EVAL, and vice versa. It is possible that MOP and EVAL are different strategies at different ends of a continuum of strategy sophistication, similar to the example of addition strategies from Siegler's research on strategy development. Alternatively, it is possible that students use these strategies differently depending on the problem characteristics (as Ainsworth suggests). To test whether strategies were different by representation pair, we used cluster analysis, a technique that allowed us to explore the relationship between strategy use and problem type or task demands.

Research question 2 asks, how are students' strategy profiles different by representation pairs in CMR tasks? To identify profiles of strategies students used to solve each CMR question type, we employed a cluster analysis (Milligan & Hirtle, 2003) on the frequency of verbalizations of strategies by each representation pair in the CMR measure. Cluster analysis is a person-centered approach that uncovers homogeneous groups underlying a set of data (DiStefano & Kamphaus, 2006), and it serves our purpose of identifying a number of strategy-use profiles (i.e., clusters). Within each cluster, participants are similar with regard to their strategy use. Between clusters, participants are distinct with regard to their strategy use from those in the other clusters.[4]

Research question 3 follows up on research question 2. It asks, if identifiable clusters by strategy are found, then do the groups fall along a continuum from less sophisticated to more sophisticated strategies for CMR? Also, is one group more successful than the other on background characteristics and related measures of mathematical proficiency? To answer both parts of research question 3 we evaluated the obtained cluster solutions by comparing the subgroups with respect to various outcomes, including accuracy in answering each type of CMR item, time used to complete each type of CMR item, other mathematics-related outcomes (i.e., arithmetic calculations during the CMR and scores on the paper-and-pencil measures), and scores on assessments of background knowledge and skills. We then interpreted the cluster analysis findings in light of the correlational results.

3. Results

3.1. Descriptive statistics

As shown in Table 3, on average students scored 84% correct on the full scale of CMR (SD = 0.17; number of items, k = 12). Mean accuracy (i.e., percentage correct) on equation-graph and graph-

[3] Note that arithmetic calculations are not a strategy, but are part of the other strategies coded above (i.e., MOP, MOPX, MOPY, MINTX, MINTY, EVAL, and CHECK).

[4] We used a hierarchical cluster analysis procedure based on squared Euclidean distances with the Ward's method on standardized scores (Milligan, 1996) of within-person proportion of strategy use (i.e., Z scores of occurrences of a particular strategy within a question type per student/occurrences of all strategies within a question type). We chose the hierarchical clustering method due to the exploratory nature of our study, where we did not have a hypothesis to support specifying a particular number of clusters which is required by a flat clustering approach (e.g., k-means). We determined the number of clusters using the dendrogram to observe the decreases in agglomeration coefficients, we also considered the cluster size, and validated our hierarchical clustering solutions with a k-means approach (Halkidi, Batistakis, & Vazirgiannis, 2001).

Table 3
Descriptive statistics of CMR measure accuracy, time used, number of strategies used, and sum scores of paper measures.

	CMR accuracy (% Correct)				CMR time spent				CMR strategies used				Paper-and-pencil measure sum scores				
	Total	EG	ET	GT	Total	EG	ET	GT	Total	EG	ET	GT	MRT	PFT	PCA	CCM	APC
M	0.84	0.86	0.81	0.85	450.9	141.0	172.1	137.9	34.3	9.1	8.4	16.9	4.7	7.0	3.8	15.8	3.1
SD	0.17	0.20	0.24	0.22	189.4	61.8	78.9	61.7	11.2	2.9	4.7	8.5	2.9	2.0	2.1	6.1	2.7
Min.	0.25	0.25	0.00	0.25	172.8	53.6	54.6	51.6	6.0	0.0	0.0	1.0	0.0	2.0	0.0	0.0	0.0
Max.	1.00	1.00	1.00	1.00	1110.3	348.7	464.8	354.9	63.0	15.0	23.0	48.0	12.0	10.0	8.0	23.8	10.0
Skew.	−1.50	−1.52	−1.52	−1.30	1.0	0.9	1.2	1.1	0.1	−0.9	0.9	1.2	0.6	−0.3	0.0	−0.7	0.6
Kurt.	2.06	1.92	2.48	0.60	2.0	0.7	2.5	1.6	−0.1	1.9	1.4	2.4	−0.2	−0.4	−1.0	−0.5	−0.3

Note. $N = 67$. $^{*}p < 0.05$. $^{**}p < 0.01$. CMR = Eye-Tracking Coordinating Multiple Representations items. EG = Equation-Graph items. ET = Equation-Table items. GT = Graph-Table items. MRT = Mental Rotation Test sum scores. PFT = Paper Folding Test sum scores. PCA = Precalculus Placement Assessment items sum scores. CCM = Calculus Conceptual Measure sum scores. APC = Advanced Placement® Calculus items sum scores.

Table 4
Bivariate correlations among CMR measure accuracy, time used, number of strategies used, and sum scores of paper measures.

	CMR accuracy (% Correct)				CMR time spent				CMR strategies used				Paper-and-pencil measure sum scores				
	Total 1	EG 2	ET 3	GT 4	Total 5	EG 6	ET 7	GT 8	Total 9	EG 10	ET 11	GT 12	MRT 13	PFT 14	PCA 17	CCM 18	APC 19
1	–																
2	0.667**	–															
3	0.868**	0.465**	–														
4	0.744**	0.248*	0.511**	–													
5	0.264*	−0.025	0.254*	0.334**	–												
6	0.222	−0.086	0.263*	0.219	0.800**	–											
7	0.258*	0.005	0.242*	0.329**	0.953**	0.638**	–										
8	0.216	−0.021	0.174	0.322**	0.900**	0.637**	0.794**	–									
9	0.448**	0.296*	0.382**	0.401**	0.650**	0.398**	0.662**	0.614**	–								
10	0.185	0.032	0.235	0.068	0.097	0.240	0.018	0.084	0.208	–							
11	0.396**	0.223	0.415**	0.328**	0.557**	0.236	0.632**	0.495**	0.747**	0.278*	–						
12	0.313**	0.258*	0.198	0.328**	0.523**	0.316**	0.523**	0.513**	0.844**	−0.216	0.346**	–					
13	0.169	0.145	0.180	0.139	0.171	0.049	0.198	0.166	0.233	0.034	0.300*	0.170	–				
14	0.270*	0.313*	0.315*	0.033	−0.009	−0.008	0.009	−0.042	0.091	−0.089	0.225	0.033	0.272*	–			
17	0.498**	0.363**	0.441**	0.401**	0.121	0.049	0.170	0.046	0.262*	0.116	0.338**	0.164	0.352**	0.262*	–		
18	0.271*	0.239	0.239	0.190	0.185	0.008	0.246	0.155	0.286*	−0.039	0.451**	0.184	0.303*	0.240	0.518**	–	
19	0.360**	0.293*	0.389**	0.152	0.100	−0.090	0.162	0.112	0.312*	−0.059	0.432**	0.244	0.352**	0.197	0.457**	0.628**	–

Note. $N = 67$. $^{*}p < 0.05$. $^{**}p < 0.01$. CMR = Eye-Tracking Coordinating Multiple Representations items. EG = Equation-Graph items. ET = Equation-Table items. GT = Graph-Table items. MRT = Mental Rotation Test sum scores. PFT = Paper Folding Test sum scores. PCA = Precalculus Placement Assessment items sum scores. CCM = Calculus Conceptual Measure sum scores. APC = Advanced Placement® Calculus items sum scores.

table questions was similar, but students scored slightly lower on equation-table questions ($M = 0.81$, $SD = 0.24$; $k = 4$ for each question type). Students used on average 450 s ($SD = 190$) to complete all 12 items, using relatively more time on equation-table items ($M = 172$, $SD = 79$) compared to equation-graph and graph-table items. Students applied more strategies (utterances of all strategies used) to solve graph-table problems ($M = 18$, $SD = 7$) than to solve the equation-graph and equation-table problems.

Students' mean scores on the paper-and-pencil measures are presented in Table 3. We did not find violation of normality for any variables of CMR accuracy, time, counts of strategies, or sum scores of paper-and-pencil measures (see Table 3 for descriptive statistics including kurtosis and skewness).

3.2. Bivariate and partial correlations

3.2.1. Full CMR

Table 4 shows bivariate correlations among accuracy, time, and verbalizations of strategy use for the full CMR scale by the 3 question types. For all 12 items on the CMR measure, accuracy is positively associated with the frequency of verbalizations of strategy use ($r[65] = 0.443$, $p < 0.001$). We also found a significant but small positive correlation between accuracy and total time used ($r[65] = 0.264$, $p = 0.031$). Importantly, even after accounting for time used, there is still a significant positive correlation between accuracy and verbalizations of strategy use ($r_{as.t}[64] = 0.377$, $p = 0.002$). These results indicate that while applying more strategies requires more time, using more strategies is associated with better overall performance on the 12 items.

3.2.2. Equation-graph items

For equation-graph items we did not find any significant correlations among accuracy, time, and verbalizations of strategy use. The partial correlation between accuracy and verbalizations of strategy use, controlling for time used, is also non-significant ($r_{as.t}[64] = 0.054$, $p = 0.666$).

3.2.3. Equation-table items

For equation-table questions, we found significant positive correlations between accuracy and time used, and between accuracy and verbalizations of strategy use ($r[65] = 0.415$, $p < 0.001$ and $r[65] = 0.242$, $p = 0.048$, respectively), but no significant correlation between time used and verbalizations of strategy use. Importantly, after controlling for time used, there is still a significant positive correlation between accuracy on equation-table questions and verbalizations of strategy use ($r_{as.t}[64] = 0.349$, $p = 0.004$). These results indicate using more strategies is associated with better performance on the equation-table questions even after controlling for time used.

3.2.4. Graph-table items

For graph-table questions, we found significant positive correlations between accuracy and time, and between accuracy and verbalizations of strategy use ($r[65] = 0.322$, $p = 0.008$ and $r[65] = 0.328$, $p = 0.007$, respectively). However, no significant correlation was found between accuracy and verbalizations of strategy use after controlling for time used ($r_{as.t}[64] = 0.200$, $p = 0.108$).

To briefly summarize findings from the correlation analyses, higher accuracy in answering equation table questions is associated with verbalizations of strategy use, but accuracy is not related to either verbalizations of strategy use or time used for equation-graph and graph-table questions. The lack of expected correlations between accuracy and verbalizations of strategy use or time used for equation-graph and graph-table questions as well as the negative correlation between the proportion of EVAL and MOP prompted us to investigate question 2, whether students were using different profiles of strategies on these items.

3.3. Cluster analysis

To further understand which of the seven coded strategies (i.e., MOP, MOPX, MOPY, MINTX, MINTY, EVAL, CHECK) may be associated with different performance on equation-graph, equation-table, and graph-table items, we used hierarchical cluster analysis to identify the strategies applied by the participants to solve each of the three types of questions.

3.3.1. Cluster analysis on strategies for equation-graph items

For the equation-graph questions on the CMR assessment, cluster analysis results indicated two distinct clusters of students who differed in use of specific strategies: cluster-1 students ($n = 49$) applied evaluation of direction, order or magnitude in about 60% of strategies coded (EVAL) and 25% of cluster-1 strategies were matching the y-intercept while explicitly referring to it (MINTY). Independent-samples t tests of accuracy, time used, and arithmetic calculations for answering equation-graph items, and scores on the paper-and-pencil measures were conducted to explore these profiles of strategy use. The members of cluster-1 used EVAL significantly more ($t[63] = 8.697$, $p < 0.001$, $d = 2.362$) compared to cluster 2 (note we report effect sizes, d. as well as p-values for each significant comparison). Cluster-2 students ($n = 16$) used all seven strategies, with significantly more MOPX (i.e., matching the x-intercept without explicitly referring to it) than cluster 1 ($t[15.9] = -5.029$, $p < 0.001$, $d = -1.726$; see Fig. 2 for strategy use profiles). We found about 9 total verbalizations of strategy use per equation-graph question by students in each cluster while solving equation-graph problems; there was no significant difference between the two clusters of students in terms of the number of verbalizations of strategy use (see Table 5; $t[63] = 0.049$, $p = 0.961$). These patterns of strategy use for equation-graph problems led us to label the two clusters *EVAL-ers* and *DO-ALL-ers.*

Comparing the two equation-graph clusters on various outcome measures, we found the members of the two clusters did not differ significantly in accuracy answering the equation-graph CMR questions ($t[63] = -0.417$, $p = 0.678$). However, the *EVAL-ers* used significantly less time than the *DO-ALL-ers* to complete the equation-graph items ($t[63] = -2.251$, $p = 0.028$, $d = -0.677$; see Table 5 for performance comparison). Thus, the use of cluster-1 strategies (60% EVAL + 25% MINTY) is associated with using less time for CMR problem solving without compromising the accuracy. Conversely, the *DO-ALL-ers* (cluster-2 students) required more time to solve the equation-graph CMR problems.

The two clusters observed on equation-graph items were similar on many other aspects of prior mathematical proficiency measured in this study, except for the percentage of arithmetic calculations in their total verbalizations of strategy use (ARI Rate) for equation-graph questions ($t[16.1] = -2.783$, $p = 0.013$, $d = -0.933$) and the Calculus Conceptual Measure scores ($t[39.0] = -2.233$, $p = 0.031$, $d = -0.579$; Table 5). The *DO-ALL-ers* articulated a higher percentage of mental arithmetic calculations than *EVAL-ers* during equation-graph problem solving. A highly plausible explanation for this difference is that mental arithmetic co-occurred with point-wise matching-representations strategies. We return to the calculus conceptual measure difference in the Discussion.

3.3.2. Cluster analysis on strategies for equation-table items

Cluster analysis results indicated two distinct profiles of strategies that students used to solve the four equation-table problems (Fig. 3). For these equation-table items the two groups of students mainly applied point matching strategies with the main difference relating the frequency of using the strategies MOP and MOPY. Recall that MOP includes matching ordered pairs that are not intercepts and MOPY includes matching the y-intercept without explicitly naming the intercept. Independent-samples t tests of accuracy,

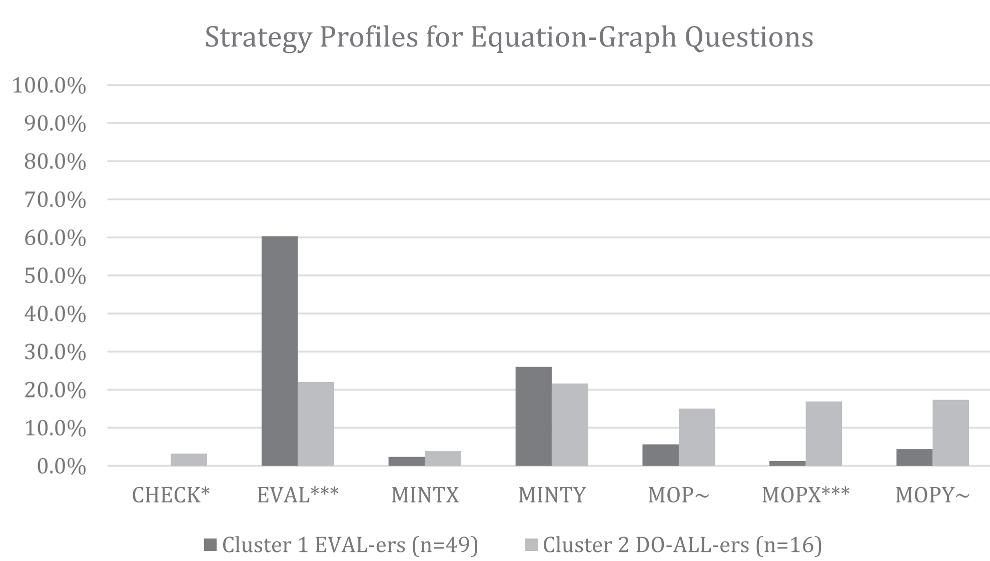

Fig. 2. Strategy profiles for the four equation-graph questions. $^{***}p < 0.001$, $^{**}p < 0.01$, $^{*}p < 0.05$, $^{\sim}$ $p < 0.10$.

Table 5
Equation-graph questions: performance by cluster.

Cluster in EG	n	Equation-graph questions M (SD)					Paper-and-pencil measure scores M (SD)				
		Strat.	Accu. rate	Time**	ARI rate*	ARIC rate	MRT	PFT	PCA	CCM*	APC
1. EVAL-er	49	9.4 (2.5)	0.85 (0.21)	136 (30)	0.04 (0.06)	0.92 (0.25)	4.8 (2.7)	6.9 (2.0)	3.6 (2.2)	15.0 (6.4)	2.8 (2.6)
2. DO-ALL-er	16	9.3 (2.3)	0.87 (0.18)	157 (32)	0.14 (0.14)	0.99 (0.02)	4.6 (3.5)	7.1 (1.9)	4.5 (1.9)	18.2 (4.3)	3.8 (2.7)

Note. Strat. = Verbalizations of strategy use. Accu. Rate = Rate of accuracy of answers. ARI Rate = Percentage of verbalizations of arithmetic calculations in total verbalizations of strategy use for a type of representation pairs. ARI-C Rate = Percentage of verbalizations of correct arithmetic calculations in all arithmetic calculations for a type of representation pairs. MRT = Mental Rotation Test scores. PFT = Paper Folding Test scores. PCA = Precalculus Placement Assessment measure scores. CCM = Calculus Conceptual Measure scores. APC = Advanced Placement® Calculus measure scores. ***$p < 0.001$, **$p < 0.01$, *$p < 0.05$, ~ $p < 0.10$.

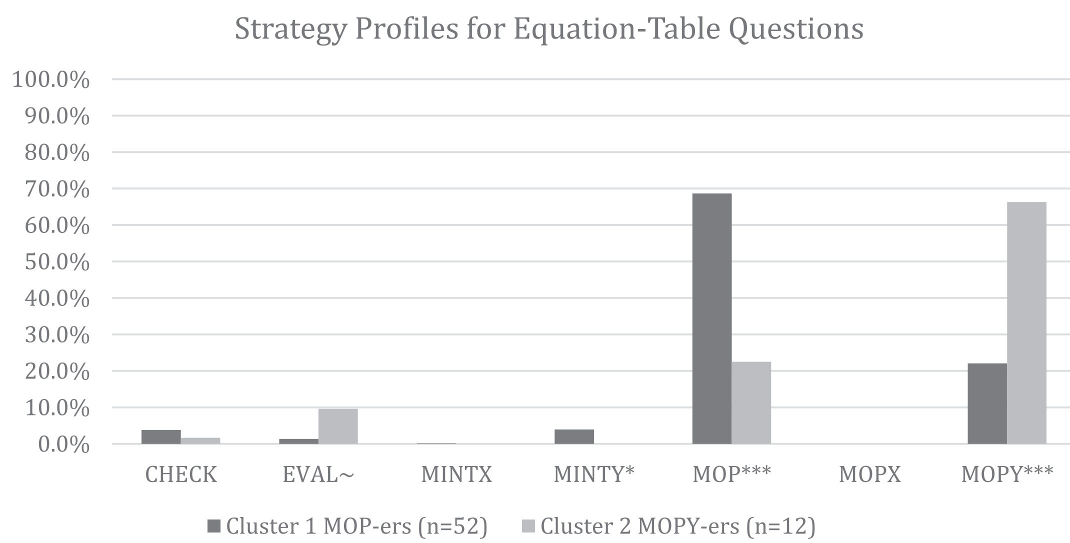

Fig. 3. Two clusters of students who used different strategies for the four equation-table questions. Note that listwise deletion was applied to handle missing data, which led to $N = 64$ for this analysis. ***$p < 0.001$, **$p < 0.01$, *$p < 0.05$, ~ $p < 0.10$.

Table 6
Equation-table questions: performance by cluster.

Cluster in ET	n	Equation-table questions M (SD)					Paper-and-pencil measure scores M (SD)				
		Strat.	Accu. rate**	Time**	ARI rate***	ARIC rate	MRT	PFT	PCA*	CCM	APC*
1. MOP-er	52	9.6 (4.3)	0.86 (0.22)	188.5 (77.1)	0.50 (0.18)	0.88 (0.15)	4.9 (2.9)	7.1 (2.1)	4.1 (2.1)	16.4 (6.0)	3.5 (2.7)
2. MOPY-er	12	5.4 (2.9)	0.67 (0.18)	110.1 (45.3)	0.27 (0.18)	0.89 (0.13)	3.5 (2.2)	6.3 (1.2)	2.7 (2.0)	13.4 (5.8)	1.3 (1.0)

Note. Strat. = Verbalizations of strategy use. Accu. Rate = Rate of accuracy of answers. ARI Rate = Percentage of verbalizations of arithmetic calculations in total verbalizations of strategy use for a type of representation pairs. ARI-C Rate = Percentage of verbalizations of correct arithmetic calculations in all arithmetic calculations for a type of representation pairs. MRT = Mental Rotation Test scores. PFT = Paper Folding Test scores. PCA = Precalculus Placement Assessment measure scores. CCM = Calculus Conceptual Measure scores. APC = Advanced Placement® Calculus measure scores. ***$p < 0.001$, **$p < 0.01$, *$p < 0.05$, ~ $p < 0.10$.

time used, and arithmetic calculations for answering equation-table items, and scores on the paper-and-pencil measures were conducted to explore these profiles of strategy use. Cluster-1 students ($n = 52$) used significantly more MOP strategies (70%; $t[62] = 8.010$, $p < 0.001$, $d = 2.693$) and fewer MOPY strategies (20%; $t[62] = -7.847$, $p < 0.001$, $d = -2.134$), whereas cluster-2 students ($n = 12$) employed significantly more MOPY (65%) and fewer MOP (20%) strategies than cluster-1 students. Note, however, that cluster-1 students also uttered significantly more total verbalizations of strategy use than cluster-2 students (10 and 5, respectively see Table 6; $t[62] = 3.187$, $p = 0.002$, $d = 1.132$).

Comparing the two clusters in answer accuracy and time used, we found the cluster-1 students, who applied 70% matching ordered pairs other than intercepts (MOP) when solving the equation-table questions had a significantly higher percentage correct in answering the equation-table CMR questions ($t[62] = 2.760$, $p = 0.008$, $d = 0.937$), though they also used more time on these questions ($t[62] = 3.378$, $p = 0.001$, $d = 1.240$; see Table 6 for performance differences). In other words, the extent of MOP as employed by cluster-1 students is positively associated with better performance in solving equation-table problems. In light of the significant positive partial correlation between strategy use and accuracy for equation-table questions, we can tell that using more strategies—doing more MOP in particular—is worthwhile in order to have better performance on equation-table questions.

The two clusters of students were similar on many other aspects of mathematical proficiency measured in this study, except for percentage of arithmetic calculations in total verbalizations of strategy use (ARI Rate) for equation-table questions (see Table 6; $t[62] = 3.894$, $p < 0.001$, $d = 1.24$), scores on Pre-calculus Placement Assessment measure scores (PCA; $t[62] = 2.201$, $p = 0.031$, $d = 0.717$), and scores on the AP Calculus measure (APC; $t[44.5] = 2.591$, $p = 0.012$, $d = 1.058$). Cluster-1 students were more accurate in equation-table questions, articulated more mental arithmetic than cluster-2 students, and scored slightly higher than cluster-2 students on the pre-calculus and AP Calculus measures.

3.3.3. Cluster analysis on strategies for graph-table

For graph-table questions, cluster analysis results indicated two clusters of students who differed in strategy use, though the cluster sizes were largely incomparable: It appears a very small cluster ($n = 6$) of students employed significantly more matching while referring to x- and y-intercepts (MINTX, MINTY) and matching ordered pairs, not including intercepts (MOP) than the other cluster. However, these two clusters did not differ in accuracy or time used to complete the graph-table items. On balance, we do not consider this 2-cluster solution practically meaningful.

3.3.4. Cluster membership overlap

Research question 3a asks whether cluster membership can be described along a continuum from less sophisticated to more sophisticated CMR strategies. In order to further examine patterns in the students' verbalizations of strategy use for equation-graph and equation-table items, we conducted a chi-square test of independence of distribution with the two clusters identified for the equation-graph items and the two clusters identified for equation-table questions. The test results indicate nonindependent distribution of equation-graph clusters (i.e., EVALers and Do-All-ers) within the two equation-table clusters (i.e., MOPers and MOPYers; $x^2[1] = 4.231$, $p = 0.027$). The EVALers, who solved the equation-graph items in less time with no sacrificed accuracy, tended to be the MOPers when answering equation-table items. Recall that on the equation-table items MOPers were more accurate than the MOPYers (see Table 7). This analysis indicates that EVALers and MOPers used more sophisticated strategies, and the non-independence of cluster membership suggests an affirmative answer to Research question 3a. Additionally the dominant strategies used by these students shifted based on the representation pair in the CMR task.

The cluster analyses show that performance on different types of representation pairs (i.e., equation-graph and equation-table) is associated with different strategies. Evaluation of direction, order, or magnitude (EVAL) appears to be associated with faster performance without compromising accuracy for answering equation-graph questions, and doing more matching of ordered pairs, not including intercepts (MOP) seems to be associated with better performance on equation-table questions, even though it takes more time. Across these two types of representation pairs, students who adopted EVAL for equation-graph questions and solved these questions more quickly were more likely to adopt MOP for equation-table questions and they solved the equation-table questions with a higher accuracy but not as quickly as the students who simply matched the y-intercepts.

Research question 3b asks whether one of the groups is more successful than the other on background characteristics and related measures of mathematical proficiency. We cannot provide a definitive answer for research question 3b because the data are mixed on whether the cluster members were more successful than the other on the other measures of mathematical proficiency or background characteristics such as spatial skills. None of the spatial skills measures differed across the clusters. For the mathematics assessments we found mixed results. One unexpected result on the Equation-Graph cluster solution was the EVAL-ers scored lower than the Do-ALL-ers on the pre-calculus (PCA) and the calculus measures (APC and CCM), and the difference in means was significant on the CCM.

Table 7
Distribution members in clusters for equation-graph items by clusters for equation-table items.

		Equation-graph items		
		EVALers	Do-All-ers	Total
Equation-Table Items	MOPers	42	10	52
	MOPYers	6	6	12
	Total	48	16	64

4. Discussion

This investigation identified seven strategies students use on CMR tasks, and used cluster analysis to explore how students' use of CMR strategies connects to both student characteristics and task demands (Ainsworth, 2006). Our research questions focused on 1) characterizing the strategies secondary students use to solve polynomial CMR tasks, 2) identifying whether strategies differed by representation pairs in the stimuli, and 3) investigating differences in the groups of students who used different strategies profiles.

We addressed research question 1 through developing the coding scheme for strategies on the CMR assessment. We identified seven strategies that students used to solve CMR tasks. Research questions 2 and 3 were investigated through statistical analysis of the study measures and strategies. This two-step process included 1) analysis of descriptive statistics and correlations between the frequency of strategy use, time, and accuracy on CMR items, and 2) cluster analysis to identify different strategy profiles on the equation-graph, equation-table, and graph-table items, followed by tests of differences in study measures by cluster. In answer to research question 2, we identified two clusters based on profiles of strategy use for the equation-graph (EVAL-ers and DO-ALL-ers) and equation-table (MOP-ers and MOPY-ers) CMR items. We were not able to identify a meaningful cluster for the graph-table items.

We found an affirmative result for research question 3a: the EVAL-ers and MOP-ers used more sophisticated strategies than their counterparts on the CMR task. A Chi Square test of group membership showed that the students who used EVAL strategies in the equation-graph condition were also more likely to be in the more successful cluster (MOP) for the equation-table representation pair. Recall the EVAL-ers solved equation-graph questions more quickly than the DO-All-ers without loss of accuracy, and the MOP-ers solved the equation-table items more accurately than the MOPY-ers.

Finally, research question 3b asked whether one group was more successful than the other on background characteristics and related measures of mathematical proficiency. We did not find a consistent pattern in results. The mixed results with the mathematics assessments for the equation-graph solutions (i.e., the fact that EVALers had non-significantly lower scores on the PCA, CCM, and APC than the Do-All-ers) is somewhat difficult to interpret. One possible explanation for this difference is that the content of the CMR assessment was distinct from, and not as difficult as, the content on the CCM or APC. Students could have developed accurate and fast CMR strategies such as EVAL before having developed calculus skills. This trend was reversed on the Equation-Table cluster solution where the MOP-ers had higher means on the pre-calculus and calculus assessments, and the difference in means was significant for the PCA and APC measures. In order to explore the relationship between strategies and different forms of content knowledge, one possible direction for future work may be to include a more heterogeneous sample, including students from non-advanced secondary mathematics courses. Similarly, the relationship between strategies and content area knowledge could be explored using assessments of less advanced secondary mathematics (e.g. algebra from early secondary grades) in order to understand strategy profiles and cluster differences more fully. Finally, that we found no differences on the background spatial skills measures between the clusters suggests that CMR strategies may be a direct target of an instructional intervention.

The cluster solutions of strategy profiles by CMR stimuli type support the research-based claim that different representations may afford different coordination strategies (Acevedo Nistal, Van Dooren, & Verschaffel, 2014; Ainsworth, 2006; Siegler, 2005). We caution that the inferences from this study are limited by both the sample and the structure of CMR prompts that were designed to prompt particular forms of correct and incorrect reasoning. The sample size and composition may help explain the unexpected results in comparing cluster solutions and results on the mathematics assessments. We conjecture that with a larger sample including students from a wider selection of secondary mathematics courses we may be able to refine our understanding of CMR strategy profiles, and the connection between strategy selection and participant's traits. Additionally, we note that limited time for our CMR assessment constrained the design of the CMR prompts. Pilot work indicated that items with mismatched y-intercepts were too easy. Therefore, all mis-matched functions on the CMR test had matching y-intercepts. However, one side effect of this choice was that the strategies MOPY and MINTY were never sufficient for identifying a mismatch. With different CMR prompts these strategies might increase in prominence.

4.1. Future directions

Overall, this analysis suggests that the representation pair in a CMR task influences the profile of CMR strategies advanced secondary students use. This finding aligns with both research on representations in mathematics (Acevedo Nistal et al., 2014; Moschkovich et al., 1993) as well as research on strategy choice (Booth et al., 2013; Jurdak & El Mouhayar, 2014; Siegler, 2005). As discussed above, one extension would be to replicate this study with a more heterogeneous sample, possibly incorporating easier mis-matched items. Another avenue for future development of this work may be to investigate how the degree of polynomial functions interacts with the representation pair and the strategy use. The number of possible permutations for representations made it impractical for us to test all possible representation pairs with all possible combinations of degree 1, 2, and 3 polynomials and all possible matches/mismatches. However, in future iterations of this work it may be possible to extend our battery of CMR assessment items to investigate how the degree of the polynomials influences CMR strategies.

The findings presented here can be used to refine models of CMR and strategy selection as it relates to more general constructs such as mathematical proficiency. For example, this work can extend the work of Wilmot, Schoenfeld, Wilson, Champney, and Zahner (2011) who validated an assessment of college readiness by examining how many connections students made while coordinating representations. In addition to the number of connections, this study and prior work on CMR suggests that students' strategies for coordinating representations can be an important target for development.

Finally, the results of this study provide some guidance for modifying CMR stimuli that might direct participants' attention to key features of representation pairs. Using this strategy framework, we can analyze whether modifications to stimuli will prompt some respondents to use accurate CMR strategies that require less time. In the long-term this can lead to interventions that will improve the quality of students' skills coordinating representations.

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