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Investigating the improvement of prospective elementary teachers' number sense in reasoning about fraction magnitude

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Abstract We report on interview results from a classroom teaching experiment in a Number and Operations course for prospective elementary teachers. Improving the number sense of this population is an important goal for mathematics teacher education, and researchers have found this goal to be difficult to accomplish. In earlier work, we devised a local instruction theory for the development of number sense, which focused on whole-number mental computation. In this study, the local instruction theory was applied to the rational-number domain, with the help of a framework for reasoning about fraction magnitude, and it guided instruction in the content course. We interviewed seven participants pre- and post-instruction, and we found that their reasoning on fraction comparison tasks improved. The participants made more correct comparisons, reasoned more flexibly, and came to favor less conventional and more sophisticated strategies. These improvements in number sense parallel those that we found previously in mental computation. In addition to the overall results, we highlight two cases of improvement that illustrate ways in which prospective elementary teachers' reasoning about fraction magnitude can change.

Keywords Prospective elementary teachers · Fractions · Number sense · Local instruction theory

Introduction

Improving the mathematical preparation of prospective elementary teachers is a pressing problem in mathematics teacher education (Conference Board of Mathematical Sciences,

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CBMS 2012). Prospective elementary teachers (PTs) often lack the understanding of elementary mathematics that they will need in order to teach that mathematics effectively (Ball 1990; Ball et al. 2008). Mathematics teacher educators have a responsibility to find effective ways of ameliorating this situation.

To help address the problem of PTs' mathematical preparation, we previously designed a local instruction theory for the development of number sense (Nickerson and Whitacre 2010). Our previous research showed that PTs involved in a classroom teaching experiment developed improved number sense, particularly in the area of whole-number mental computation (Whitacre and Nickerson 2006). In a recent iteration of the classroom teaching experiment, the local instruction theory was extended from the whole-number to the rational-number portion of a Number and Operations course. An established written measure of number sense (Hsu et al. 2001) was administered pre- and post-instruction to 34 students enrolled in the course. Consistent with the results of our previous study, we found that the participants' number sense scores improved. We interviewed a subset of the students to hone in on their reasoning about fraction magnitude. In comparing their responses to fraction comparison tasks pre- and post-instruction, we found that participants' performance improved, they became more flexible, and they shifted to favoring less conventional and more sophisticated strategies. These changes parallel those that we found previously in mental computation.

In this article, we first describe the rationale for our research. We present the local instruction theory and describe how it was extended to the rational-number domain. We describe the interviews, and we present results that speak to changes in participants' reasoning about fraction magnitude. We further describe two cases of number sense improvement. One case highlights a PT becoming more flexible in her reasoning about fractions. The other focuses on a shift in a PT's reasoning about fractions that enabled her to make sense of and use nonstandard strategies. Thus, this paper contributes to the research literature in three ways: (1) by documenting an instance of the successful improvement of PTs' fraction sense, (2) by describing the design of instruction for the course in which these results occurred and relating that design to a broader local instruction theory, and (3) by characterizing the nature of the productive changes that were seen in PTs' reasoning about fraction magnitude.

Background

This study derives from an ongoing design research effort (Cobb et al. 2003) concerned with improving prospective elementary teachers' number sense. According to McIntosh et al. (1992).

> Number sense refers to a person's general understanding of number and operations along with the ability and inclination to use this understanding in flexible ways to make mathematical judgments and to develop useful strategies for handling numbers and operations. It reflects an inclination and ability to use numbers and quantitative methods as a means of communicating, processing and interpreting information. It results in an expectation that numbers are useful and that mathematics has a certain regularity. (p. 3)

The development of number sense has been a priority in mathematics education in the USA and internationally (Australian Education Council 1991; DfEE 1999; Japanese Ministry of Education 1989; National Council of Teachers of Mathematics, NCTM 2000; National

Research Council, NRC 2001). Teachers of elementary students need strong number sense in order to foster their students' development of mathematical proficiency, including conceptual understanding, productive disposition, adaptive reasoning, and strategic competence (NRC 2001). The Common Core State Standards for Mathematics (CCSSM) describe students who are able to use benchmark numbers and their understanding of magnitude to estimate and assess the reasonableness of answers involving whole numbers and fractions. The Standards for Mathematical Practice suggests students should make sense of problems, reason about quantities and the relationship among them, understand how problems can be solved in more than one way, and look for and express regularity in reasoning (National Governor's Assoc. 2010). Engaging students in these mathematical practices aligns well with an emphasis on number sense. In order to support their students' engagement in problem solving and their development of number sense, teachers need improved number sense themselves (CBMS 2012; Ma 1999; Sowder 1992; Yang et al. 2009). In particular, a teacher whose understanding of numbers and operations is bound to standard algorithms is not equipped to make sense of children's often-unorthodox solution strategies (Ball 1990; Ball et al. 2008; Philipp 2008).

Fractions are a particularly challenging topic in elementary mathematics, both for students and for prospective and practicing elementary teachers (e.g., Ma 1999; National Center for Education Statistics, NCES 2014; National Mathematics Advisory Panel 2008; Newton 2008). On the 2007 National Assessment of Educational Progress, 50 % of 8th graders could not order fractions from least to greatest (NCES 2014). The official recommendations of the Institute of Education Sciences (IES) include extending instruction on fractions to ordering and equivalence and providing opportunities for students to locate and compare fractions on a number line, which is cited as a central representational tool in teaching fraction concepts (Siegler et al. 2010). Understandably, another recommendation with regard to fractions is the need for "improving teachers' understanding of fractions and how to teach them" (Siegler et al. 2010, p. 42). The study of PTs' knowledge of fractions is important because fractions, in particular, are "notoriously difficult to learn and to teach" (Newton 2008, p. 1082).

In this section, we set the stage for the present study with a brief review of relevant literature and a description of our approach to instructional design.

Design research

Design research aims at both achieving learning outcomes and building theory (Gravemeijer 1999, 2004; Cobb et al. 2003). Our previous research involved the design and elaboration of a local instruction theory for students' development of number sense (National Governor's Association and the Council of Chief State School Officers 2010). The previous classroom teaching experiment focused on the topic of whole-number mental computation. In the recent classroom teaching experiment, the local instruction theory was applied to reasoning about fraction magnitude—another microcosm of number sense.

Researchers have found that prospective elementary teachers tend not to reason about fractions in meaningful, sophisticated ways, even after having completed college mathematics courses (Tsao 2005; Yang 2007; Yang et al. 2009). A previously reported instructional intervention was unsuccessful in promoting improvement in prospective elementary teachers' fraction sense (Newton 2008). In particular, in the context of an instructional intervention focused on improving PTs' knowledge of fractions, Newton found that the PTs' fraction knowledge improved in some respects; however, their flexibility in reasoning about fractions did not improve.

The present study investigated whether and how PTs' fraction sense could improve in the setting of a Number and Operations course. Specifically, we asked, *as prospective elementary teachers participate in an instructional sequence on fraction magnitude, informed by a local instruction theory for the development of number sense, how does their reasoning about fraction comparisons change?* In design research, theory informs practice, and practice feeds back to inform theory (Gravemeijer 1999). The empirical aspects of our larger research involved documenting collective classroom activity and analyzing student learning through student interviews and written work. This report focuses on the results of our analysis of interview data.

Previous research

One area of focus in the number sense literature has been the computational strategies that students use in solving problems involving operations with whole numbers. Better number sense is associated with more flexibility, which is exhibited in the use of a variety of computational strategies. In mental computation, inflexibility often manifests in the use of the mental analogues of the standard paper-and-pencil algorithms. While the standard algorithms can be useful, people who exhibit number sense in mental computation tend to select a strategy for an operation based on the particular numbers at hand, rather than using a single go-to strategy. Furthermore, the strategies that these individuals use often stray far from standard, as in reformulating computations or rounding and compensating (Carraher et al. 1987; Greeno 1991; Heirdsfield and Cooper 2002, 2004; Hope and Sherrill 1987; Markovits and Sowder 1994; Reys et al. 1982, 1995; Sowder 1992; Yang et al. 2009). Reformulating computations or rounding and compensating rely on an understanding of place value, benchmark numbers, and properties of operations. For example, in reformulating 47–29 to 47–30 in order to compute the difference, one must recognize the possibility and the efficacy of the reformulation, as well as the proximity of 29 to 30, subtract correctly, and then compensate appropriately.

In order to teach mathematics effectively, elementary teachers need to understand elementary mathematics deeply, be able to analyze students' responses, and use appropriate models in explanations (Ball 1990; Ball et al. 2008). However, researchers investigating prospective elementary teachers' number sense have found them to reason inflexibly about numbers and operations, even after having completed their undergraduate mathematics coursework (Newton 2008; Tsao 2005; Yang 2007; Yang et al. 2009). In light of these findings, our research has focused on improving PTs' number sense.

In 2005, we conducted a classroom teaching experiment in a Number and Operations course for prospective elementary teachers. In that study, we focused on mental computation as a microcosm of number sense. Instruction was guided by a local instruction theory for the development of number sense, which we describe in the next section. Thirteen students participated in interviews pre- and post-instruction, in which they were given story problems to be solved mentally. In addition to coding for the particular strategies that participants used, we coded these as belonging to more general categories of strategies. We used a scheme of Markovits and Sowder (1994) to categorize mental computation strategies as *Standard*, *Transition*, *Nonstandard*, and *Nonstandard with Reformulation*. The essential criterion in this scheme is the extent to which the person's approach is tied to (or departs from) the standard algorithm for a given operation. Nonstandard strategies are those that diverge substantially from the standard algorithms. The use of such strategies suggests an understanding of the operation and of number composition that is not bound to any particular algorithm. As such, the use of nonstandard strategies is associated with number sense (Markovits and Sowder 1994; Yang et al. 2009).

The Standard-to-Nonstandard framework revealed a rather dramatic shift in the strategies used by the 13 interview participants. In the first interview, standard strategies were most common. Participants used few strategies for the operations and often relied on the mental analogues of the standard written algorithms. In the second interview, by contrast, the most common category of strategy was nonstandard with reformulation—the other extreme of the continuum. Thus, participants shifted from favoring the most standard to the least standard strategies, which suggests that their understanding of the operations moved from being bound to the standard algorithms to being unconstrained by these (Whitacre 2007). These results were encouraging and led us to pursue further research concerning the development of PTs' number sense.

Local instruction theory

The previous teaching experiment was reflexively related to the development of a local instruction theory. A *local instruction theory* (LIT) consists of "the description of, and rationale for, the envisioned learning route as it relates to a set of instructional activities for a specific topic" (Gravemeijer 2004, p. 107). In a previous article, we describe our LIT for the development of number sense with a focus on whole-number mental computation (Nickerson and Whitacre 2010). Broadly, the envisioned learning route involves students moving from dependence on standard algorithms to reasoning flexibly about numbers and operations. More specifically, our LIT is organized around three goals: (1) students capitalize on opportunities to use number-sensible strategies for problem-solving situations both inside and outside the classroom; (2) students draw on deep, connected knowledge of number and operations to develop a repertoire of number-sensible strategies; and (3) students reason with models to build on their understanding and flexibly use number-sensible strategies (Nickerson and Whitacre 2010). The envisioned learning route is one in which students move from relying on standard algorithms to developing and using their own nonstandard strategies. Students invent, justify, and name their strategies, using models such as reasoning about the distance between numbers on the number line to collectively make sense of the nonstandard strategies. Students reason with models to flexibly draw on and extend a repertoire of number-sensible strategies.

Applying the LIT to reasoning about fraction magnitude

Originally, our LIT was developed with a focus on whole-number mental computation. In the 2010 study, we sought to extend it to the rational-number domain with a focus on students' reasoning about fraction magnitude. Like mental computation, this area is rich with opportunities for PTs to exercise their number sense. There are traditional procedures for comparing fractions, which are often taught in school, as well as a variety of possible nonstandard strategies. Behr et al. (1984) touted the importance of reasoning about fraction magnitude as a prerequisite to reasoning meaningfully about operations involving fractions. If prospective elementary teachers are to reason meaningfully about division of fractions, for example, they must first be adept at comparing the magnitudes of the fractions involved. Other researchers, such as Yang and colleagues, have used fraction estimation and comparison tasks in assessments of students' number sense (Hsu et al. 2001; Reys and Yang 1998; Yang 2007).

Reys et al. (1999) describe number sense as consisting of six components: (1) understanding of the meaning and size of numbers, (2) understanding and use of equivalent representations of numbers, (3) understanding the meaning and effect of operations, (4) understanding and use of equivalent expressions, (5) flexible computing and counting strategies for mental computation, written computation, and calculator use, and (6) measurement benchmarks. In our work on whole-number mental computation, we focused especially on Component 5, and we viewed the development of flexible computing strategies as supported by Components 3 and 4. In designing the fraction magnitude unit, we focused especially on Component 1 with support from Component 2. As Behr et al. (1984) argue, reasoning meaningfully about fraction magnitude is a prerequisite for reasoning meaningfully about operations involving fractions (Components 3, 4, and 5).

We selected fraction comparison tasks as occasions for reasoning about fraction magnitude. It is important to note that fraction comparisons can be accomplished without reasoning about fraction magnitude. A prime example of this is the standard algorithm of converting two fractions to equivalent forms with a common denominator and then comparing the numerators, as in comparing 4/5 with 5/6 by converting to 24/30 and 25/30. Once the conversion is accomplished, only knowledge of whole-number magnitude is required to make the comparison (i.e., $24 < 25$). Ironically, this method enables one to effectively compare the magnitudes of two fractions without necessarily reasoning about the magnitude of either of them.[1] In terms of promoting the development of PTs' fraction sense, a major goal was to encourage our students to actually reason about fraction magnitude.

Smith's (1995) framework informs our thinking concerning students' reasoning about fraction magnitude, especially in the context of fraction comparison tasks. Smith groups fraction comparison strategies into four categories, which he calls *perspectives*. These perspectives serve not only to categorize strategies but also to highlight commonalities in reasoning across groups of strategies.

Strategies such as converting to a common denominator or converting to a decimal belong to the *Transform* perspective. They involve transforming one or both fractions in some way in order to facilitate the comparison. For example, a student may compare 3/4 and 4/5 by using division to convert these fractions to decimal form and then determine which is greater by comparing 0.75 with 0.8.

Fractions can also be compared without performing transformations. One way to do this is to apply the *Parts* perspective, wherein the fractions are interpreted in terms of parts of a whole. This perspective is sufficient to reason through comparisons in some cases, such as when comparing fractions that have the same numerator or same denominator. For example, 1/7 is greater than 1/8 because partitioning a whole into fewer equal-sized parts (7 vs. 8) creates larger parts. It follows that 3/7 is greater than 3/8 because three of a larger amount is greater than 3 of a smaller amount.

The *Reference Point* perspective involves reasoning about fraction magnitude on the basis of proximity to reference numbers—also known as *benchmarks* (Parker and Leinhardt 1995). Reference Point reasoning relates to the number line. For example, to compare 7/8 and 6/7, a student may notice that 7/8 is 1/8 away from 1, whereas 6/7 is 1/7 away from 1. He

[1] We acknowledge that finding a common denominator can be a useful approach to fraction comparison tasks. We further acknowledge that this procedure can be performed with conceptual understanding and can work hand in hand with reasoning about fraction magnitude. However, we find that this procedure is widely used but rarely understood.

or she then compares the *residuals*, or distances from 1. Since a distance of 1/8 is less than a distance of 1/7, 7/8 is closer to 1 and therefore larger.

The *Components* perspective involves making comparisons within or between two fractions, as in coordinating multiplicative comparisons of numerators and denominators. For example, in order to compare 13/60 and 3/16, we can notice that $13 \times 5 = 65 > 60$, whereas $3 \times 5 = 15 < 16$. It follows that 13/60 is greater since its numerator–denominator ratio is less extreme.

In designing instruction, these perspectives informed our decisions relative to tasks, number choices, and anticipated student reasoning. We mapped out the envisioned learning routes described in our LIT in terms of the evolution of these perspectives and of particular strategies within each category.

We view the Reference Point and Components perspectives as generally more sophisticated categories of reasoning about fraction magnitude. There is support for this view in the literature. For example, Yang (2007) considers the Residual Strategy (within the Reference Point perspective) to be number sense based, as opposed to rule based. We posit that there is a general correspondence between Smith's perspectives and the Standard-to-Nonstandard framework, described earlier. In particular, the Transform and Parts perspectives correspond more or less to the Standard and Transition categories of strategies, while the Reference Point and Components perspectives correspond to Nonstandard strategies (with or without reformulation). We do not intend by this a one-to-one mapping of categories, but a more general grouping into Standard (including Transition) and Nonstandard. Smith distinguishes between *instructed* and *constructed* strategies for comparing fractions, based on the degree to which he found support for these in popular textbooks. He found little support for Reference Point or Components strategies in texts. Smith's constructs of instructed and constructed align well with the categories of Standard and Nonstandard.

The general number sense literature, as well as specific studies by researchers such as Yang (2007) and Newton (2008), suggested that PTs would come to the mathematics content course with limited number sense and would tend to apply standard algorithms for comparing fractions. Pilot interviews that we conducted with prospective elementary teachers who had completed their mathematics content courses confirmed this expectation: even after having completed their content coursework, the pilot interview participants displayed very limited reasoning about fraction magnitude. In our instructional sequence, we aimed for the more sophisticated strategies to eventually be used by students and established for the class by mathematical argumentation. In particular, we sought to engage students in reasoning about fraction magnitude from the Reference Point and Components perspectives. Tasks were designed and sequenced so as to begin with students' current ways of reasoning and to provide opportunities for reasoning about fractions in new ways. Classroom norms were negotiated with the intention that students would develop autonomy in judging the efficacy and validity of the strategies. The instructor viewed student-invented strategies as resources that helped to promote the learning of the class.

Hypothetical learning trajectory

Following the process described above, the general LIT for number sense development informed a specific hypothetical learning trajectory (HLT) focused on reasoning about fraction magnitude. An HLT consists of a learning goal, a set of learning activities, and associated conjectures regarding how students will reason as they engage in the activities. It thus presents a vision of how learning may proceed (Simon 1995). Below, we describe the Number and Operations course and the fraction magnitude sequence in more detail. We lay out our learning goal, sequence of learning activities, and conjectures regarding progress in PTs' reasoning.

The course was intended to foster the development of number sense. In particular, the broad instructional intent was for students to come to act in a *mathematical environment* (Greeno 1991) in which the properties of numbers and operations afford a variety of computational strategies, as opposed to one in which mathematical operations map directly to particular algorithms. Topics in the curriculum begin with quantitative reasoning (Smith and Thompson 2007), then understanding place value, meanings for operations, children's strategies, standard and alternative algorithms, representations of rational numbers, and operations involving fractions. Our work in the classroom teaching experiment involved identifying in the curriculum particular opportunities to engage students in such activities as mental computation and reasoning about fraction magnitude, as well as to facilitate rich discussions concerning students' strategies and ways of reasoning. Over time, a shared set of strategies was established via mathematical argumentation. These strategies were given agreed-upon names, and the class maintained a list of strategies with examples of each. The class utilized both area models and the linear model of the number line in justifications of students' strategies.

Students came to the course with a Parts conception of fractions, readily relating fractions to "pies" or to rectangles. Parts reasoning and an area model served as a foundation upon which more sophisticated strategies came to be established. Students also engaged in activities involving placing fraction markers on a string representing a number line. Distance along the number line came to feature prominently in students' fraction comparison arguments.

More specifically, the HLT involves the following sequence of learning activities:

- In early discussions of fraction comparisons, students are encouraged to make their prior knowledge explicit.
- The class establishes specific meanings for fractions and reinforces foundational ideas, such as the inverse relationship between the number of parts that a whole is divided into and the size of those parts.
- Students are challenged to relate Parts reasoning to familiar Transform procedures, such as finding a common denominator. Students use area models to justify Parts reasoning and Transform procedures.
- Once basic Parts comparisons are established in the common numerator and common denominator cases; students are invited to build on these to solve nontrivial comparisons, such as 5/19 with 15/59.
- Deliberately chosen numbers, such as 5/6 and 6/7, invite students to recognize complements (the Parts analogue to residuals) and to compare their magnitudes.
- Moving on from comparing fractions, students are asked to place sets of fractions in specific positions on a number line, while attending to their relative order. Numbers are chosen to afford comparisons of residuals.
- Numbers are chosen to afford comparisons to benchmarks other than 1, such as 1/2, 1/4, 1/3, and 2/3. Tasks are designed with three possible cases in mind: straddling a benchmark, comparing distances below a benchmark, and comparing distances above a benchmark.
- Students are invited to reflect on their recognition of benchmarks and to engage in discussions focused on how they recognize that a fraction is close to 1/2, or 1/4, or 2/3.

- Students are encouraged to move beyond the number line to draw conclusions regarding relative magnitudes based on Components reasoning alone.

As the class moved through this sequence of learning activities, we conjectured that students' reasoning would develop from instructed toward constructed strategies.

More specifically, we expected that students would first rely on Transform knowledge, especially converting to a common denominator (Ma 1999; Newton 2008), and they would begin the unit with somewhat limited Parts knowledge (Yang 2007; Yang et al. 2009). As they participated in class activities, we expected that students would develop more robust Parts knowledge by reasoning about fractions as quantities (Tzur 1999). Parts reasoning would support making comparisons by attending to complements. In number line activities, reasoning about complements would support a transition to comparing fractions to benchmarks. Finally, students would begin to engage in multiplicative Complements reasoning by attending to numerator-to-denominator ratios as a way of recognizing proximity to benchmarks.

Methods

This study took place at a large, urban university in the southwestern USA. The participants in the study were students enrolled in a first mathematics content course for prospective elementary teachers, belonging to a four-course sequence. There were 39 students enrolled in the course, and 38 of them were female. Most students were in their first year of college. The second author was the instructor of the course. She was a mathematics educator and an experienced instructor of mathematics courses for prospective teachers. There are multiple sections of the course, and a common final examination is used.

As a specific measure of number sense, an English language version of the Number Sense Rating Scale (Hsu et al. 2001) was administered pre- and post-instruction to 34 of the students (A few students were absent on one day or the other of test administration). The NSRS is a multiple-choice test, which was originally designed to assess the number sense of middle school students. It and similar instruments have been used to assess the number sense of both middle school students and preservice elementary teachers (e.g., Reys and Yang 1998; Tsao 2005; Yang 2003). The NSRS includes tasks involving rational numbers, as well as tasks that are restricted to whole numbers. The majority of the tasks involve fractions and/or decimals. Written work is not allowed on the NSRS. If any evidence of written work was seen on students' pages, the related items were marked as incorrect. Students were allowed 20 min to complete the NSRS. This amount of time was sufficient for the participants to complete the survey.

There are 37 items on the NSRS, which are counted as correct or incorrect with a value of one point per item. Thus, the maximum score is 37 points. Pretest scores ranged from 11 to 35 points (30–95 %). The mean score on the pretest was 24 points (65 %), compared to a mean score on the posttest of 29.4 points (79 %). This improvement in scores was statistically significant ($p < 0.0001$), and the effect size was greater than 1 standard deviation (Cohen's $d = 1.27$). Partitioning the pretest scores into three categories—below 22, from 22 to 27, and above 27—yields approximately equal-sized groups. On the pretest, there were 11 students who scored in the low group, 11 in the middle group, and 12 in the high group.

Seven students volunteered and participated in pre- and post-instruction interviews involving fraction comparison tasks. The interview participants were all female undergraduates. We refer to the interview participants by the pseudonyms Angela, Brandy, Zelda, Natalie, Trina, Maricela, and Valerie. Table 1 presents the interview participants'

Table 1 Interview participants' NSRS scores and gains compared to groups

Student	Pre-score	Group	Gain (post–pre)	Mean gain for group
Angela	18	Low	13	9.36
Brandy	21	Low	8	9.36
Zelda	23	Middle	9	4.27
Natalie	24	Middle	8	4.27
Trina	26	Middle	4	4.27
Maricela	28	High	5	2.92
Valerie	29	High	3	2.92

NSRS pretest scores, which we identify as belonging to the low, middle, or high group. The table also shows the participants' gain score (post–pre), as well as the mean gain score for her group. Comparing the pretest scores of the interview participants to those of the whole class, there were two from the low group, three from the middle group, and two from the high group. Thus, as measured by the NSRS, the number sense of the interview participants at the beginning of the course varied from low to high relative to the class.

Interview participants were asked to evaluate the relative sizes of pairs of fractions. Nine pairs of fractions were presented, one at a time. A list of the pairs of fractions that were used appears in Fig. 1 (left). The same pairs of fractions were used in both interviews. The fractions were presented visually on a strip of paper, and participants were asked to read them aloud. Participants solved the tasks mentally and explained their reasoning verbally, as shown in Fig. 1 (right). Participants were not allowed to do any written work for this portion of the interviews.

A team of researchers worked to modify the framework of Smith (1995) for use as a coding scheme (see the "Appendix"). It needed to be modified due to differences in student population,[2] the particular set of tasks that we used, and the purposes of our research. For example, Smith did not use any improper fractions in his comparison tasks, so the strategy of converting these to mixed numbers needed to be added to the scheme. There were also strategies that we removed from the scheme because they never occurred in our data set, and there were pairs of strategy codes that we collapsed into a single code. The refined scheme was used to code the pre- and post-instruction data. We also coded strategies as Standard or Nonstandard, as discussed earlier.

In addition to coding for strategies and perspectives, we coded responses as Valid or Invalid. These codes refer to the researchers' assessment of the mathematical validity of a participants' strategy. Fraction comparisons are multiple-choice tasks with only three options (one fraction is larger, the other fraction is larger, or the two are equal). As a result, it is not unusual for students to give correct answers on the basis of invalid or unclear approaches. We were interested in identifying *valid strategies that led to correct responses*. Thus, we also coded responses for correctness. As an assessment of change in performance, we compared the number of Valid and Correct (VC) responses pre- and post-instruction for each interview participant. To assess change in flexibility, we compared the number of distinct valid strategies used in VC responses by each participant pre- and post-instruction. Finally, we compared the numbers of Standard and Nonstandard VC responses pre- and post-instruction.

[2] Smith's (1995) original framework is based on the reasoning of K-12 students.

2/8	3/8
3/4	3/5
6/7	7/8
14/13	13/12
8/24	13/39
13/60	3/16
7/28	13/50
2/7	12/43
35/832	37/834

Fig. 1 (*Left*) list of fraction comparison tasks; (*right*) Maricela responding to a task

Results

In this section, we present results that speak to changes in flexibility and overall performance on the comparison tasks. We present two cases of change, one that highlights improved flexibility and one that highlights the adoption of valid strategies. Finally, we present results that illustrate how participants' reasoning shifted from predominantly Standard strategies to a more balanced range of Standard and Nonstandard strategies.

Improved performance and flexibility

For six of the seven interview participants, the numbers of VC responses (correct answers obtained by the use of a mathematically valid strategy) increased from the first to the second interview. The mean of such responses increased from 5.86 to 7.7 (of a total of 9 responses). The additional VC responses often coincided with new strategies used by the participants. Invalid strategies that led to incorrect responses were replaced by valid strategies that led to correct responses. The variety of distinct, valid strategies used in VC responses also increased for six of the seven participants. The mean number of distinct strategies used increased from 4.86 to 7.57. Thus, participants used valid strategies more often in the second interview, *and* they used a wider variety of valid strategies (see Table 2).

The data in Table 2 reveal different kinds of changes. For example, Angela gave only four VC responses in the first interview, whereas she gave eight in the second interview. Maricela, by contrast, showed little change in VC responses. However, Maricela used only three distinct strategies in her VC responses in the first interview, whereas she used eight distinct, valid strategies in VC responses in the second interview. Thus, her ability to get correct answers by valid strategies did not change much; however, her flexibility in responding to fraction comparison tasks improved substantially. Below, we elaborate on these two cases of change.

Maricela's transformation from inflexible to flexible

Improved flexibility was a general trend across the interview participants. However, the change observed in Maricela was especially profound. Maricela was particularly inflexible in her reasoning about fraction magnitude in the first interview. For seven of the nine

Table 2 Total VC responses and distinct valid strategies in VC responses pre- and post-instruction

Student	Total VC responses		Distinct valid strategies	
	Pre	Post	Pre	Post
Angela	4	8	4	7
Brandy	4	7	4	7
Maricela	7	8	3	8
Nancy	8	9	6	9
Trina	7	8	6	8
Valerie	5	8	5	8
Zelda	6	6	6	6
Mean	5.86	7.7	4.86	7.57

comparison tasks, she attempted to solve by finding a common denominator. In some cases, this approach was manageable and led her to the correct answer. However, she also attempted this approach on comparisons tasks for which it was extremely unwieldy, including the comparison of 35/832 and 37/834 (Maricela spent more than 10 min attempting to mentally compute 832 × 834, 35 × 834, and 37 × 832. She was unsuccessful). Such conspicuous inflexibility suggests that her reasoning about fraction magnitude was bound to the standard algorithm of finding a common denominator by multiplying denominators. Even though pencil and paper were unavailable to her, she entertained no alternative other than attempting to mentally compute and retain multiple multi-digit products.

In her second interview, Maricela used a distinct strategy for each of the nine comparison tasks, and eight of these were VC responses. In her first interview, she had compared 13/60 with 3/16 by converting to a common denominator. In her second interview, she compared these by comparing their distance from 1/4. She noted that 3/16 is 1/16 away from 1/4, while 13/60 is 1/30 away from 1/4. She used this information to conclude that 13/60 was the larger of the two fractions:

> "3/16 is close to 4/16, and it's just 1/16 away. So, I did the same for this one [13/60]… So, 13/60 is 2/60 away, which also is equal to 1/30. So, I compared 1/16 and 1/30… 13/60 is bigger because 1/16 is the gap between 3/16 and 4/16. So, the gap is bigger than the gap of 1/30, which is smaller, so 13/60 is bigger."

In comparing 13/60 with 3/16, Maricela used a particularly sophisticated version of a logically and conceptually complex strategy. We coded her primary strategy as *Distance Below* under the Reference Point perspective. She identified a benchmark fraction that would be useful for comparison. This required an initial estimation of the size of the fractions as being roughly close to that benchmark of 1/4. She then identified fractions equivalent to 1/4 but with denominators of 60 and 16, which facilitated her distance comparison by making it easy to find differences. She identified the distances from 1/4 as 2/60 and 1/16. She simplified 2/60 to a unit fraction, which then enabled her to compare the "gaps" of 1/16 and 1/30. She stated that 1/16 was the larger of these (The reason for this was not made explicit, but we know from her response concerning 3/4 and 3/5 earlier in the interview that she is capable of justifying this kind of comparison from a Parts perspective). She then correctly concluded that 13/60 was larger than 3/16 because it was closer to 1/4. The reasoning that Maricela displayed here contrasts starkly with her very procedural approach to this and other fraction comparison tasks in her first interview.

Maricela's one incorrect response in the second interview came in comparing 35/832 and 37/834. She noted the common difference of 2, comparing the numerators and denominators of the pair of fractions (Components Additive Between), and she concluded that the two fractions were equal. While the strategy that she attempted for this very challenging comparison task was invalid, we noted progress in the direction of improved number sense as evinced by the fact that Maricela noticed a relationship in the given numbers and attempted to make use of it. This is in contrast to her first-interview response in which she made no apparent choice based on the numbers, instead laboring through an extremely inefficient approach.

Angela's adoption of valid strategies

The change seen in Maricela had primarily to do with flexibility. The change observed in other students, like Angela, involved exchanging invalid strategies for valid ones, as she came to reason differently about fraction magnitude. In her first interview, Angela reasoned about the size of parts but focused heavily on the denominators. On the 9 fraction comparison tasks, Angela gave 5 incorrect answers, and 4 of these were attributed to Denominator Dominance. She accurately compared the sizes of "slices" or "equal sections" of a "pie" but did not adequately account for the different numerators, or numbers of such slices.

To illustrate, when asked to compare 8/24 with 13/39 in her first interview, Angela reasoned, "The 39 is larger, and so it's gonna make the pieces *really* small." She concluded that 8/24 was greater without explicitly accounting for the numerators in any way. To compare 35/832 with 37/834, she said, "Each slice is smaller, so even though 37 are being filled instead of 35, the pieces are *so* small that 35 out of 832 looked larger." Angela focused on how the denominator influenced the size of the pieces, and this single aspect dominated her thinking. By contrast, in her second interview, Angela accounted for both numerators and denominators in her reasoning. For example, she compared 6/7 with 7/8 by reasoning about how much was "missing" from the whole, and she correctly concluded that 7/8 was greater. When thinking about 35/832 versus 37/834, Angela reasoned that the pieces were approximately the same size, so *the number of pieces* would make more of a difference than size in this case.

Angela also accounted for comparisons of both numerators and denominators by invoking multiplicative Components reasoning between. She compared 3/16 and 13/60 by reasoning about numerator-to-denominator ratios. She reasoned that $3 \times 5 = 15$, which is less than 16, whereas $13 \times 5 = 65$, which is greater than 60. She noted that 15/16 was less than 1 and 65/60 was greater than 1. She concluded that 13/60 was larger.

In the first interview, Angela thought about fractions in terms of parts of a whole, but she was unable to adequately coordinate comparisons between numerators and denominators, except in the special cases of a common numerator or denominator. She initially focused on denominators and made her comparisons as though the numerators were equal, even when they were not. By contrast, in her second interview, Angela successfully accounted for both numerators and denominators by using more sophisticated Parts and Components strategies.

Shift from standard to nonstandard

Maricela's contrasting responses pre- and post-instruction suggest a shift in perspectives. Her responses to eight of the nine comparison tasks in the first interview reflected a

Transform perspective. In particular, she used Convert to Common Denominator for seven of the comparison tasks, Reduce to Lowest Terms once, and Parts Numerator Principle once. Thus, she tended to approach comparison tasks by applying a procedure for converting one or both fractions to an equivalent form in order to make the comparison. In her second interview, by contrast, Maricela's responses reflected a range of perspectives. Four of her nine responses were coded as Parts, two as Reference Point and two as Components, and there was only one instance of the Transform perspective.

This shift in perspectives was also a trend across the interview participants. In the first interview, 30 of the 41 VC responses involved a Standard strategy, such as converting to a common denominator. In the second interview, by contrast, there were 54 VC responses, and 30 of these involved Nonstandard strategies. These data are presented in Fig. 2. We note that the responses in the first interview were weighted largely toward Standard. This imbalance reflects the lack of flexibility exhibited by many of the interview participants. In the second interview, responses involving Nonstandard strategies outnumbered those involving Standard strategies, but by a relatively small margin. This more balanced set of strategies used is a desirable picture from our perspective. There is nothing wrong with the Standard strategies in principle. In fact, the research team agreed that four of the nine comparison tasks that were posed lent themselves well to Standard approaches. Thus, we would expect flexible, skilled individuals to use such approaches approximately as frequently as the interview participants did for the given set of tasks. In fact, exactly 24 of 54, or 4/9, of the VC responses were Standard.

Figure 2 provides a concise summary of the interview results: We see the shift from Standard to Nonstandard, the increase in VC responses demonstrates improved performance, and the increase in Nonstandard VC responses coincides with improved flexibility.

Shifts in perspective/strategy on particular items

To further examine the variety of strategies that participants used, we focus on particular items. We saw substantial change ($|post - pre| \geq 2$) in the frequency of VC responses on 5 of the 9 comparison tasks (see Table 3). It was on the items listed in Table 3 that we observed noteworthy shifts in participants' perspectives and strategies used. These changes in strategies are summarized in Table 4, which enumerates instances of valid strategies in correct responses, categorized by perspective.

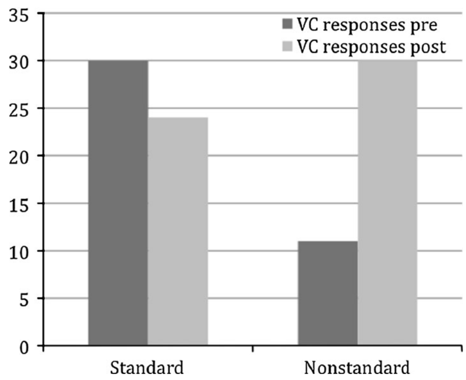

Fig. 2 Shift from Standard to Nonstandard strategies

Table 3 Instances of VC responses by item

Item	VC responses		
	Pre	Post	Change
6/7_7/8	2	7	+5
13/60_3/16	4	6	+2
7/28_13/50	4	6	+2
2/7_12/43	3	6	+3
35/832_37/834	1	3	+2

Table 4 Perspective counts for VC responses by item

Item	Perspectives pre				Perspectives post			
	Trans.	Parts	RP	Comp.	Trans.	Parts	RP	Comp.
6/7_7/8	1	1			1	6		
13/60_3/16	2			2	1		3	2
7/28_13/50	3		1				4	2
2/7_12/43	2			1	6			
35/832_37/834				1		2		1

When comparing 6/7 with 7/8 in the first interview, four participants made an invalid Parts Denominator Dominance comparison and two transformed to a common denominator, one correctly and one incorrectly. In the second interview, by contrast, six participants correctly compared complements—a valid strategy that is particularly well suited to the fractions involved in this task. Another difference in the participants' reasoning on these tasks was the relative prevalence of Reference Point strategies. The Reference Point perspective appeared very rarely in the first interview. It was not used to compare 13/60 with 3/16 and was used only once to compare 7/28 with 13/50. In the second interview, Reference Point strategies were used three times to compare 13/60 with 3/16 and were used four times to compare 7/28 with 13/50. As with comparing complements, the Reference Point strategies were well suited to the particular tasks for which they were used. Participants compared 13/60 and 3/16 to either 1/4 or 1/5. They also recognized that 7/28 was equivalent to 1/4, and they compared 13/50 to 1/4. The shift to more Transform strategies for comparing 2/7 to 12/43 reflects the increase in use of the strategy of transforming to a common numerator in the second interview. That pair of fractions does not lend itself well to the standard procedure of finding a common denominator. However, transforming to a common numerator works nicely for those numbers. The task of comparing 35/832 with 37/834 was difficult for participants in both interviews. In the second interview, there were more instances of VC responses, and these involved Parts and Components reasoning.

Conclusion

In previous research, we found that prospective elementary teachers' whole-number sense improved when instruction in a Number and Operations course was guided by an LIT for number sense development. In this study, we extended the LIT to the rational-number domain and investigated changes in PTs' reasoning about fraction magnitude. Smith's (1995) framework informed our thinking about strategies and perspectives involved in comparing fractions. The research literature, our previous experience with students in the content course, and our pilot interviews enabled us to anticipate PTs' initial reasoning about fraction magnitude. Our pre-instruction interviews enabled us to refine our expectations further. We designed the instructional sequence on the basis of our students' anticipated starting points, conceived in terms of Smith's framework as essentially the Transform and Parts perspectives, and with the goal of building toward the Reference Point and Components perspectives. Specific instructional activities were then crafted with this broad envisioned learning route in mind.

Results from pre- and post-instruction interviews with seven participants provide evidence that their performance and flexibility in comparing fractions improved. Furthermore, the participants shifted from using predominantly Standard strategies in the pre-instruction interview to a more balanced set of both Standard and Nonstandard strategies in the post-instruction interview. Thus, there is evidence that the participants developed improved fraction sense. These results also parallel those that we saw in whole-number mental computation in our previous research. With respect to both the whole-number and rational-number domains, we have found the LIT to be helpful in guiding instructional decisions, and we have seen evidence of growth in PTs' number sense. The fact that these results have been achieved with PTs is especially significant since the literature indicates that PTs are particularly in need of number sense, and these kinds of results are not typical (e.g., Newton 2008; Tsao 2005; Yang 2007; Yang et al. 2009).

Whereas the PTs in Newton's (2008) study did not become more flexible in their reasoning about fractions, we saw evidence that our participants' flexibility did improve. In particular, 6 of the 7 interview participants adopted at least two new valid strategies for comparing fractions. Maricela's was the most extreme case of improved flexibility, as she used five new valid strategies in the post-instruction interview. Angela's case illustrated how a PT's reasoning about fraction magnitude can change as she comes to coordinate comparisons of both the numerators and the denominators in comparing fractions. Overall, the participants performed better on fraction comparison tasks, scoring on average nearly two more correct responses post-instruction versus pre-instruction. They also used a wider variety of valid strategies to perform those comparisons, demonstrating improved flexibility in reasoning about fraction magnitude.

Consequently, findings from the study provide evidence of the kind of change that is possible and offer mathematics teacher educators guidance in supporting PTs to reason meaningfully and flexibly about fraction magnitude. The intervention that we have described suggests the inclusion of comparison tasks and placement tasks on the number line, as well as particular sequencing of these and other tasks; this instruction can result in improved reasoning in terms of flexibility and sophistication. Both the IES report (Siegler et al. 2010) and the CCSSM (2010) stress the centrality of the number line as a representational tool in teaching fraction concepts. We can begin with PTs' current Parts reasoning, which they can build upon to make sense of Transform procedures. By using tasks with well-chosen numbers, we can again leverage PTs' Parts reasoning and understanding of equivalence to recognize Complements and transition to reasoning about residuals. As PTs are invited to compare fractions to benchmark numbers other than 1, they compare numerators to denominators using Components reasoning.

It is important for these comparison tasks to be embedded in a classroom culture in which number sense is valued and strategies are an explicit topic of conversation, so that PTs can develop autonomy in navigating mathematical environments. Asking prospective teachers to use a number line, to invent their own comparison strategies, and to construct arguments about fraction magnitude aligns well with the emphasis in the CCSSM on engaging students in mathematical practices, such as looking for and making use of structure, using appropriate tools strategically and constructing viable arguments and critiquing the reasoning of others (National Governor's Assoc. 2010). Successful and flexible reasoning about fraction magnitude is an important prerequisite for reasoning meaningfully about operations involving fractions (Behr et al. 1984). Mathematics teacher educators can provide these opportunities for reasoning about fraction magnitude in the context of an extant curriculum.

It is beyond the scope of this article to make recommendations regarding fraction instruction for students in the elementary grades. Our instructional sequence is guided by an envisioned learning route in which students move from relying on standard algorithms to making sense of an using nonstandard strategies. It does not apply to students whose initial reasoning is different from that which is typical of PTs, and it has not been tested with K-12 students. We note that development of the perspectives and strategies identified by Smith (1995) is not clearly articulated in the CCSSM. Thus, we encourage further research focused on supporting children in learning to reason meaningfully about fraction magnitude. In particular, it would be useful to identify opportunities in existing curricula to support each of the perspectives and to map out a viable instructional sequence and associated trajectory with explicit connections to the standards. We also encourage further research focused on how middle-grades teachers can apply their understanding of fraction magnitude to classroom instruction.

This study addresses the important problem of improving prospective elementary teachers' mathematics content knowledge (CBMS 2012). It is encouraging to see PTs with a range of initial NSRS scores adopt more valid strategies and become more flexibility in their reasoning about fraction magnitude. They will be better equipped to lead classrooms of students who make sense of mathematics if they themselves reason in ways that are less bound to standard algorithms. At the same time, much more remains to be investigated. Further research is needed to investigate whether similar results will occur with different groups of PTs. More could be learned by interviewing larger numbers of PTs. Furthermore, this research could be extended to study changes in PTs' reasoning about arithmetic operations involving fractions.

Appendix: Fraction comparison strategies[3]

Transform perspective

In general, we code strategies as *Transform* when one or both fractions are converted to an alternate form and the student's activity/explanation appears to be procedural, as opposed to being grounded in a Parts interpretation.

Reduce to Lower Terms. The student compares the two fractions by first reducing one or both of them to lower terms. If the transformation results in the fractions being identical,

[3] This scheme is a revised version of that of Smith (1995, pp. 45–47). In the presentation here, we organize fraction comparison strategies by perspective, and we order the perspectives in accord with the hypothetical learning trajectory: Transform, Parts, Reference Point, and Components. Note that this ordering applies broadly on the level of the perspectives. Naturally, the emergence of particular strategies does not follow a strict ordering.

the student concludes that they are equal. If the two fractions reduce to different fractions, the student compares these by some other strategy, such as the Numerator Principle.

Raise to Higher Terms. The student compares the two fractions by first raising one or both of them to higher terms. If the transformation results in the fractions being identical, the student concludes that they are equal. If the transformation results in different fractions, the student compares these by some other strategy, such as the Denominator Principle.

Convert to Common Denominator. The student compares two fractions by transforming one or both so that they have a common denominator. The student may then employ the Numerator Principle or some other strategy to make the comparison.

Convert to Common Numerator. The student compares two fractions by transforming one or both so that they have a common numerator. The student then employs the Denominator Principle to make the comparison. If the student applies a Parts interpretation, the perspective should be coded as *Parts.*

Cross Multiply. The student compares two fractions by first multiplying the numerator of each fraction by the denominator of the other fraction. If these products are equal, the student concludes that the fractions are equal. If the products are not equal, the student selects the fraction whose numerator was a factor in the greater product as the larger of the two fractions.

Convert to Decimals. The student compares two fractions by converting them to decimal form. The conversion may be accomplished by long division, recall, or some other method. The student then compares the decimal numbers to determine which of the given fractions is greater.

Convert from Improper to Mixed. The student compares two improper fractions by first transforming both of them to mixed numbers. The student then employs a different strategy to compare the remaining fractional parts of the two mixed numbers (In our data, mixed numbers were between 1 and 2).

Parts perspective

Numerator Principle. Given a comparison in which the denominators are equal, the student selects the fraction with the greater numerator as the larger of the two fractions. The student's explanation addresses the fact that the denominators are equal and applies a Parts interpretation.

Denominator Principle. Given a comparison in which the numerators are equal, the student selects the fraction with the lesser denominator as the larger of the two fractions. The student's explanation addresses the fact that the numerators are equal and applies a Parts interpretation.

Compare Complements. The student compares two fractions by comparing their complements and concludes that the fraction with the smaller complement is greater. The student's explanation makes applies a Parts interpretation and makes explicit the logic that a smaller complement implies a greater fraction.

Denominator Dominance. Given a comparison in which neither the numerators nor the denominators are equal, the student selects the fraction with the lesser denominator as the larger of the two fractions. The student's explanation applies a Parts interpretation to the comparison of the denominators but does not account for the fact that the numerators are not equal.

Numerator Dominance. Given a comparison in which neither the numerators nor the denominators are equal, the student selects the fraction with the greater numerator as the larger of the two fractions. The student's explanation applies a Parts interpretation to the comparison of the numerators but does not account for the denominators being unequal.

Combination. In some cases, students combine the strategies above. For example, a student might compare 7/9 with 6/11 by noting that ninths are larger than elevenths and that seven is more than six, so that 7/9 is greater. This strategy essentially combines the Numerator Principle and the Denominator Principle into a two-part argument. We code such a strategy as Parts Combination and note the particular strategies that were combined.

Reference Point perspective

Straddle. The student compares two fractions by reasoning that one fraction is greater than a benchmark fraction and the other is less than that same benchmark fraction. The student concludes that the fraction that is greater than the benchmark is the greater of the two fractions.

Distance Below. The student compares two fractions by comparing them to a benchmark fraction. In cases where both fractions are less than the benchmark, the fraction that is closer to the benchmark is the larger of the two fractions. A special case of Distance Below occurs when proper fractions are compared to 1. We code this as the *Residual Strategy.*

Distance Above. The student compares two fractions by comparing them to a benchmark fraction. In cases where both fractions are greater than the benchmark, the fraction that is further from the benchmark is the larger of the two fractions.

Components perspective

Larger Components. The student compares two fractions by choosing the one with larger components as greater (e.g., 5/8 > 2/3 because 5 and 8 are larger numbers than 2 and 3).

Additive Within. Given a pair of proper fractions, the student compares the fractions by comparing the differences between the numerator and denominator in each. The student selects the fraction with the smaller numerator–denominator difference as the greater fraction. If the differences are equal, the student concludes that the fractions are equal.

Additive Between. Given a pair of proper fractions, the student compares the fractions by comparing the differences between the numerators (one numerator minus the other) and the differences between the denominators. If the differences are equal, the student concludes that the fractions are equal (We only saw this strategy employed in the case of a common difference).

Multiplicative Within. The student compares two fractions by comparing the numerator–denominator ratios in each. The student need not explicitly discuss ratios, but necessarily makes multiplicative comparisons between the numerators and denominators. In cases of proper fractions, the student concludes that the fraction with the greater numerator–denominator ratio is greater. If the ratios are equal, the student concludes that the fractions are equal. Multiplicative comparisons involved in this strategy may be exact or approximate.

Multiplicative Between. The student compares two fractions by comparing the numerator–numerator and denominator–denominator ratios. The student need not explicitly discuss ratios but necessarily makes multiplicative comparisons between numerators and between denominators.

Iteration. The student compares two fractions by comparing multiples of them, using the same multiplier for both. For example, 3/16 is compared to 13/60 by multiplying both fractions by 5, which yields 15/16 and 65/60. The student concludes that the fraction that results in a greater product is the greater of the two fractions.

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