Journal for Research in Mathematics Education
2013, Vol. 44, No. 5, 809–850

Students' Mathematical Noticing

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Even in simple mathematical situations, there is an array of different mathematical features that students can attend to or notice. What students notice mathematically has consequences for their subsequent reasoning. By adapting work from both cognitive science and applied linguistics anthropology, we present a focusing framework, which treats noticing as a complex phenomenon that is distributed across individual cognition, social interactions, material resources, and normed practices. Specifically, this research demonstrates that different centers of focus emerged in two middle grades mathematics classes addressing the same content goals, which, in turn, were related conceptually to differences in student reasoning on subsequent interview tasks. Furthermore, differences in the discourse practices, features of the mathematical tasks, and the nature of the mathematical activity in the two classrooms were related to the different mathematical features that students appeared to notice.

Key words: Algebraic reasoning, Discourse practices, Linear functions, Noticing

Mathematical situations often present an overabundance of information, visual cues, and possible patterns, making it impossible to process everything at once. Humans cope with this problem in a number of ways—one way is through attending to only some of the input, an act that we refer to as *noticing*. Specifically, we define noticing as selecting, interpreting, and working with particular mathematical features or regularities when multiple sources of information compete for one's attention. For example, even a seemingly simple table of linear data (as shown in Figure 1) is information dense. One student may notice that consecutive *y*-values have a constant difference of six. Another may notice a pattern in the differences between *y*- and *x*-values (e.g., 0, 4, 8, 12). Yet another student may

This research is supported by the National Science Foundation under grant REC-0529502. The views expressed do not necessarily reflect official positions of the National Science Foundation. We express thanks to Ricardo Muñoz for his efforts in data collection and analysis and to Dr. Janet Bowers for creating the Geometer's Sketchpad sketch used in the research. Drafts of a subset of the findings reported here were presented at the 2008 AERA conference and the 2007 IMBES conference. A second major finding from the same data set was presented at the 2009 AERA conference and the 2009 PME conference. We thank Jaime Diamond, Kathleen Heid, and the reviewers for their helpful feedback.

notice that each x-value is one third the corresponding y-value. What these students notice has significant ramifications for how they reason about linear functions. Simply put, “What you do not notice, you cannot act upon” (Mason, 2002, p. 7).

x	y
0	0
2	6
4	12
6	18
8	24

Figure 1. Data representing a linear function.

When student noticing has been tackled in mathematics education research, it has not been the primary area of emphasis. Specifically, research on the generalization of patterns has identified a tendency for students to notice recursive rather than functional relationships (Lannin, Barker, & Townsend, 2006; Lee, 1996; Rivera & Becker, 2008). However, rather than foregrounding noticing as an object of systematic analysis, these researchers have concentrated their investigations on how students express and justify the general case of a pattern using algebraic notation. Research on the use of computer technology in mathematics education has reported differences in what students notice versus what experts notice (Bowers, 1997; Goldenberg, 1988; Lee & Lee, 2009). However, the central concerns of such research have been the affordances and constraints of technology, the identification of design principles, and the nature of student learning with technology. Research on the use of intuitive rules has examined the saliency of different perceptual features of tasks, such as the area versus perimeter of geometric figures, and its relationship to different problem-solving strategies (Babai, Levyadun, Stavy, & Tirosh, 2006; Tirosh & Stavy, 1999; van Dooren, de Bock, Weyer, & Verschaffel, 2004). However, the emphasis in this work has been on the extent and boundaries of using a small set of intuitive rules to explain a variety of misconceptions and alternative conceptions.

On the other hand, a critical mass of researchers have begun to describe their work as being about *teacher noticing* (National Council of Teachers of Mathematics, 2010; Sherin, Jacobs, & Philipp, 2010). Central to this research is the image of a mathematics teacher bombarded with a cacophony of input, who attends selectively to some of the data (Sherin & Star, 2010). For example, one teacher may notice behavioral problems, another may attend to the gender of students volunteering during whole-class discussion, while another teacher may notice the quality of mathematical explanations that students provide. The literature on teacher noticing suggests that what teachers notice has ramifications for their pedagogical responses (Erickson, 2010; Sherin & Star, 2010; Sherin & van Es, 2009). A similar understanding of the consequences of noticing for students’ subsequent mathematical reasoning has yet to emerge but is one aim of our research.

Readers may wonder why we distinguish students' mathematical noticing from their mathematical reasoning as an arena for special investigation when we could treat noticing as an aspect of reasoning. Singling out noticing can provide researchers with a macro-level lens that highlights important aspects of learning missing from micro-level accounts of the evolution of students' reasoning. For example, consider the detailed account of a class of sixth graders learning to reason about the motion of an elevator from a speed–time graph (Noble, Nemirovsky, Dimattia, & Wright, 2004). This fine-grained research documents students' incremental changes as they learn to: (a) comprehend a graph as providing information about an elevator's motion, (b) recognize counting in the graph to make sense of speed, and (c) associate the negative region of the graph with the downward motion of the elevator. However, adopting the lens of noticing might provide important additional information to researchers and teachers, specifically that these students had not noticed a second quantity (namely time) by the end of the instructional unit. The absence of time from the students' consideration could constrain the type of subsequent reasoning that is possible. For example, coordinating two quantities to form a ratio is not possible without noticing both quantities. In short, the macro-level lens of noticing prompts us to ask simple questions, such as "Which and how many quantities do students notice?" This, in turn, can raise awareness of whether or not students have selected the features of mathematical situations that are most crucial as a foundation for particular mathematical ideas.

In this article, we use the macro-level lens of noticing to investigate the consequences of what students notice mathematically during classroom instruction and to explain how student noticing emerges and shifts. We pose two research questions:

1. How do students in different classrooms studying the same mathematics content differ in what they notice mathematically, and what are the consequences of such differences for students' reasoning on subsequent tasks?
2. What aspects of the classroom practices contribute to the emergence of and shifts in student noticing?

In asking these questions, we assume the epistemological stance that mathematical noticing is a phenomenon that is distributed across individual cognition, social interactions, material resources, and normed practices.

Mathematical Noticing as a Distributed Phenomena

Although noticing has not been a major organizing construct in the research on student learning of mathematics, it has received explicit attention in transdisciplinary research. Our perspective draws upon research on noticing as an individual cognitive phenomenon from cognitive science and neuroscience and on noticing as a socially situated phenomenon from applied linguistic anthropology. Research in cognitive science has identified three types of attention with neurological correlates: alerting, orienting, and executive attention (Fan, McCandliss, Sommer, Raz, & Posner, 2002; Posner & Petersen, 1990). *Executive attention* involves selecting one dimension or piece of information in the presence of competing sources of information and is similar to what we mean by noticing. Distinguishing executive attention from *alerting* (preparing for a sensory signal) or *orienting* (turning toward a sensory signal) helps us differentiate our work from research that examines the effect of whether or not students are "paying attention." Cognitive researchers across a variety of domains—reading, economics, Chinese, and memory concepts in psychology—have investigated how interventions using attentional treatments, contrasting cases, and systematic variation are related to gains in learning outcomes and are attributed to productive noticing by learners (Chik & Marton, 2010; Marton & Pang, 2006; McCandliss, Beck, Sandak, & Perfetti, 2003; Schwartz & Bransford, 1998).

We build upon this research by investigating the mathematical features that are salient to individual students in information-dense situations. However, we extend the notion of executive attention by conceiving of noticing as rooted in the learning process of *reflective abstraction* (Campbell, 2001/1977; Goodson-Espy, 2005). Specifically, noticing can be captured by the initial aspects of reflective abstraction, such as re-presenting mental records of one's activities, isolating characteristics of mathematical events, and identifying regularities among the records related to the learner's goals (von Glasersfeld, 1995). Grounding noticing within a learning process does not mean that the main goal of our research is to generate fine-grained learning trajectories. Instead, we take a more macro-level view in identifying the major mathematical features that students notice during class and then account for the ramifications of this noticing on students' reasoning on subsequent tasks. However, by connecting noticing to reflective abstraction, we posit that what one notices mathematically can serve as the rootstock upon which one constructs ways to reason in new situations. Several different shoots (specific ways of reasoning) may be supported, but each is constrained by the rootstock (what one notices).

Noticing is not just an individual cognitive process. According to Goodwin (1994), "The ability to see a meaningful event is not a transparent, psychological process, but is instead a socially situated activity" (p. 607). Goodwin demonstrated how people from different professions—lawyers, police, and archaeologists—view particular events through perceptual frameworks that are developed within those professions. For example, to account for how novice archaeologists come to see the color of dirt at a dig site, Goodwin situated this task in the community's overarching goal (using differences in dirt color to make inferences regarding where objects such as post holes once resided), and he identified the available material resources (i.e., using a trowel to select a sample of dirt, applying water to the sample, and then placing the sample under holes cut into a Munsell Color Chart). Goodwin further argued that the Munsell Color Chart represents a specific discourse practice, called a *coding scheme*, in which categories of meaning are located in a larger organizational system than the individual. The chart serves as a means to categorize the color of the dirt according to encapsulated solutions developed by past archeologists, thus providing "an historically constituted architecture for perception" (p. 609).

In a more dramatic demonstration, Goodwin showed how a police expert in the 1992 trial of the Los Angeles police officers charged with the beating of

Rodney King shaped what features jurors noticed in a videotape recorded by an eyewitness. The expert capitalized on the affordances of the material resources (replaying episodes from the videotape and freezing the action) and regularly employed the discourse practice of a coding scheme to name King's movements using a category from the police profession (e.g., the expert labeled slight movements of King's arms or legs as "aggression"). Goodwin's account is situated within the activity of arguing a legal case and therefore helps demonstrate how what the jurors came to notice in the video contributed to their verdict of an acquittal.

We build upon Goodwin's work by using the particular discourse practices that he identified as well as by situating the emergence of what students notice within the tasks, artifacts, and participatory structures of the classroom. However, we also make two adaptations. First, Goodwin's work emphasized perceptual noticing, likely due to the contexts he examined (e.g., archeologists categorizing the color of dirt, jurors watching a video). Because our investigation involves constructs that are not directly observable (e.g., ratios), we follow Mason (1998) by including both perceptual features and conceptual entities as things one can notice. To illustrate, reconsider the table of data from Figure 1. A student can orient her gaze to the 2 and 6 in the second row but notice at least two mathematical relationships—an additive comparison of 4 (how much greater than 2 is 6) or a multiplicative comparison of 3 (how many times as great as 2 is 6). Furthermore, a student may join the 2 and 6 into a composed unit (Lamon, 1995), bring forth the new entity as salient, and then operate on the unit (by splitting and iterating) to generate new entries for the table such as (1, 3) and (3, 9).

Second, in Goodwin's example, the jurors were not permitted to interact verbally with the police expert, a condition not typically found in classrooms. The few studies on students' mathematical noticing have also focused on communication and attention-focusing actions in one direction. For example, Stevens and Hall (1998) tracked the moves that a tutor made to train what a student noticed, such as placing her hands to physically block part of a graph. Similarly, our research group's previous research identified the ways in which aspects of instructional environments acted upon students to direct their attention (Ellis & Grinstead, 2008; Lobato, Ellis, & Muñoz, 2003). For this study, we have developed the *focusing framework*, which we present in the next section, in part to support a dynamic and reflexive account of the contributions of both students and teachers to social interactions that are related to the emergence of what students notice.

The Focusing Framework

With the focusing framework, we offer both a conceptualization of noticing and a methodological guide for the analysis of student noticing within classroom data. As stated above, this conceptualization is grounded in noticing being a distributed phenomenon. In particular, the four-component focusing framework accounts for what individuals notice as well as how noticing is shaped by the "interactions among people and between people and other material and informational systems in their environments" (Hatano & Greeno, 1999, p. 647). Each component of the framework—centers of focus, focusing interactions, mathematical tasks, and the nature of mathematical activity—is briefly characterized in turn.

First, *centers of focus* are the properties, features, regularities, or conceptual objects that students notice. Specifically, students select particular aspects of a mathematical situation when many sources of information compete for their attention. However, because this process is not directly accessible, we infer what students notice from their verbal reports, gestures, and written inscriptions. Because we define centers of focus as what individuals notice, we expect to see multiple centers of focus at play when we examine any given classroom of students.

The next three components of the framework are used to help account for the emergence of centers of focus. Goodwin (1994) asserted that what is noticed arises through the interplay between a set of discourse practices and the domain of scrutiny being employed within a specific activity. Similarly, we argue that student noticing emerges through the interplay between a set of discourse practices called focusing interactions and features of mathematical tasks during engagement in particular types of mathematical activity.

Focusing interactions refer to the discourse practices (including gesture, diagrams, and talk) that give rise to particular centers of focus. For example, Goodwin (1994) identified two discourse practices that help explain how noticing is socially organized, namely highlighting and coding schemes. Highlighting refers to visible operations upon external phenomena, such as labeling and annotating, which can shape the perceptions of others by making particular features prominent. A coding scheme refers to the use of a category of meaning by a professional as a lens through which to view events (e.g., a police expert labeling a policeman's kicks as "tools"). We discuss highlighting and coding schemes in detail and apply them to the domain of mathematics later in the Results section.

Mathematical tasks form the backdrop for discourse practices because these are the situations that students and teachers discuss. As such, the features of tasks can influence what students notice mathematically. For example, a task that asks students to draw diagrams of real-world linear functions affords an opportunity for students to notice relationships between measureable aspects of the situation as opposed to tasks that only support opportunities to notice number patterns (Ellis, 2007).

The *nature of the mathematical activity* present in a classroom refers to the participatory organization that establishes the roles governing students' and teachers' actions and that contributes to the emergence of centers of focus (P. Cobb, personal communication, September 10, 2006). Whether or not students are expected to create new strategies, explain their reasoning, and respond to their classmates' ideas can influence the number and nature of the candidates for centers of focus. The norms governing who is to provide mathematical content and how tightly interactions are guided by the teacher also can influence what students notice. Whereas focusing interactions refer to the specific discursive moves of teachers or students that serve to direct the attention of others to particular features mathematically, the nature of mathematical activity refers to the global character of discourse practices that regulate who is allowed to talk and what types of contributions they can make.

The way the framework could be used to capture the complexity of noticing in the classroom and the types of relationships that may exist among the components of the framework are illustrated in Figure 2. In the situation depicted in Figure 2, three centers of focus (represented by the target-like circles on the left) emerge as students work on a mathematical task (represented by the octagon on the left). Three students (represented by three black dots) notice a particular mathematical property or feature; four students notice different mathematical information; and two students notice a third aspect of the mathematical content. Students' centers of focus typically shift and converge over time (as represented by the arrows pointing to two different centers of focus on the right). Methodologically we find it useful to analyze student and teacher contributions to focusing interactions (represented by the grey ellipses in Figure 2) that occur during the time in which a center of focus shifts. The framework situates these focusing interactions in the nature of the mathematical activity present in the classroom (represented by the rounded rectangle) and in the mathematical tasks (represented by octagons), which provide the context for classroom engagement. The situation depicted in Figure 2 is simplified; students can notice more than one mathematical feature in a given task, and a single focusing interaction can give rise to multiple centers of focus. However, the figure illustrates that the focusing framework is an interactional system in which no one element serves as the sole explanation for a phenomenon; rather, multiple components function together to bring about particular centers of focus.

Methods

Setting and Participants

The study reported in this article was part of a larger research project funded by the National Science Foundation titled *Coordinating Social and Individual Aspects of Generalizing Activity: A Multi-Tiered Focusing Phenomena Study*. The overarching goal of the project was to understand how the generalization of

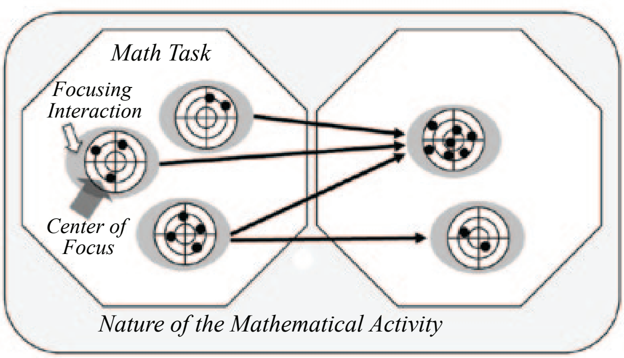


Figure 2. Use of the focusing framework in a hypothetical situation.

learning experiences is related to attention-focusing and to account for both what individuals notice mathematically in classrooms, as well as the social organization of noticing. This goal was accomplished through a series of empirical studies addressing a variety of mathematical content. The aim of the study reported in this article was to investigate how contrasting centers of focus could emerge across different classrooms of students studying the same mathematics content (linear functions) and how differences in what students noticed might be related to how they generalized their learning to reason on novel tasks after instruction.

Students for this study were recruited from a middle school of 1,334 students in a large southwestern urban school district in the United States with an ethnically mixed population (about 70% Hispanic and Filipino combined). The demographics for the students involved in this study were similar to those of the school. Our goal was to recruit 32 seventh graders in order to form four different classes of eight students each, who would meet after school for 10 hours of instruction on linear functions (taught on 10 consecutive school days for 1 hour per day). To accomplish this goal, we presented the study to all students in five regular math classes at the middle school (300 students). Thirty-four seventh graders returned consent forms, and all participated in the study. Classes 1 and 3 had eight students each; Classes 2 and 4 had nine students each.

We used a blocked random assignment of each student to one of four classes, based on results from a screening test of classic tasks related to content foundational to linear functions, such as rates and proportions (Allain, 2000; Ben-Chaim, Fey, Fitzgerald, Benedetto, & Miller, 1998; Kaput & Maxwell-West, 1994). Although the tests were scored as low, medium, or high, there was a floor effect on the screening test, indicating that the students entered the study with weak proportional reasoning. While the sample size is not large enough to support strong causal claims, our use of randomization provides some protection against selection bias and helped us meet our goal of forming four small classes of students of matched ability.

We used a teacher recruitment procedure designed to maximize contrasts in what teachers would likely emphasize when teaching about linear functions. Additionally, we recruited teachers from outside the participating students' middle school to minimize any perceived risk that participating in the study would adversely affect students' grades in their regular math classes or their reputation among teachers at their school. From a pool of 25 teachers who were contacted by the first author, eight teachers agreed to participate in a screening process, which included writing a lesson plan on linear functions and engaging in an 80-minute semistructured videotaped interview (Ginsburg, 1997). During the interviews, teachers also responded to student thinking related to linear functions, reacted to excerpts from hypothetical teachers' lessons on slope, and elaborated their goals for a unit on linear functions. On the basis of these data, we selected four teachers who demonstrated a competent grasp of the mathematical content (meaning that they did not make any mathematical errors when responding to the interview questions) but who each appeared to conceive of different aspects of the content as being important for their students (e.g., emphasizing slope as a rate of change, as the measure of the steepness of physical objects such as mountains or stairs, or as a value obtained from the slope formula).

After recruitment, the four teachers were asked to develop ten 1-hour lessons for a group of eight or nine seventh graders of mixed ability. To control for mathematics content, the teachers were asked to address the same content goals: (a) develop an understanding of slope as a rate of change; (b) informally explore linear and nonlinear functions; and (c) emphasize connections across tabular, graphical, and algebraic representations of linear functions. However, the teachers were given freedom to design their own lessons or use any existing curricular materials that they wished. Because the recruited teachers incurred travel costs going to and from the school where the lessons were taught (which was the middle school that supplied the participating students) and because we estimated that lesson planning would take approximately 40 hours, each teacher was given an honorarium of $1,000.

Central to the aim of this study was the need for at least two classrooms in which different mathematical foci emerged for the same content. However, we believe that it is very difficult (if not impossible) to engineer targeted mathematical foci in classrooms with fidelity. Consequently, we collected data in four afterschool classes in order to increase the likelihood of obtaining two classes with different mathematical foci. The data from Classes 1 and 2 ended up meeting our criteria, and we present the results of analyses of these data in this article.

Both teachers for Classes 1 and 2 used reform-oriented curricular materials and real-world settings. Teacher 1 used the context of speed throughout the unit, while Teacher 2 started with growing visual patterns and later presented the context of steepness as it relates to staircases and hills. Teacher 1 developed her own materials, while Teacher 2 drew upon curricular materials by Fulton and Lombard (2000) with some supplementation of her own activities. To help ensure that differences in reasoning on interview tasks across classes could be attributed to differences in the instructional environments rather than to students' knowledge prior to instruction, the study was conducted before the students had instruction on linear functions or proportions in their regular mathematics classes. In the participating school district, these topics were introduced for the first time in Grade 7 (after our study concluded).

Data Sources and Data Reduction

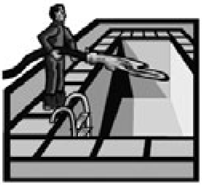
To enable an investigation of what individual students noticed mathematically during instruction, we studied smaller afterschool classes. By using three camcorders, each recording the discussion of one of three small groups, we captured each student's verbalizations and gestures during group work. During whole-class discussions, one of the three camcorders was turned to capture the verbal utterances, gestures, and written work of the teacher and the students who presented at the white board located in the front of the classroom. Because bodies could block the camcorders from recording boardwork, still photos of inscriptions placed on the white board were also taken. Finally, students' activity sheets were collected.

A 60-minute, videotaped, semistructured interview (Bernard, 1988) was conducted for each student within a week of the last session of the student's 10-hour class. Three of the research team members (two of the authors and a research assistant) served as interviewers. All interviews occurred after school. Because all research team members had engaged with the students before and after the videotaping of the instructional sessions, there was a level of comfort and familiarity during the interviews. The interview protocol was designed to address the core mathematical content covered during instruction but was set in four contexts that none of the classes had investigated (i.e., water pumped into a swimming pool, orange juice mixtures, arcade game pricing, and a burning candle situation). The same tasks were presented to each student, but the follow-up questions were tailored to individuals to probe their reasoning (Ginsburg, 1997). Students were given the choice to talk as they solved a problem or to describe their reasoning after reaching a solution. Because these were not teaching interviews, the interviewer took care not to introduce new mathematical language or ideas, and students were encouraged to come to their own conclusions about any mathematical questions that they had.

It was crucial to reduce the vast amount of interview and classroom data to a manageable amount with a narrowed focus for analysis (Miles & Huberman, 1994). If we could identify major differences in how students reasoned about linear functions in the interviews, then these would be candidates for the consequences of student noticing. Limiting the interview findings further to the presentation of one major difference in students' reasoning across classes would still permit us to answer the research questions but would, in turn, allow for space to present the results of the four classroom analyses called for by the focusing framework.

We immediately reduced the interview data to the tasks set in the water-pumping context because these took about half the interview time and seemed to evoke rich responses from students (according to the interviewers). Then research team members created a descriptive log (see Miles & Huberman, 1994) of what a subset of students from each class did to answer each task in the water-pumping context without yet engaging in the inferential interpretation that marks our analysis (as described in the Methods of Analysis section). As a result, we identified four potential themes as a target for analysis (following Bowen, 2006): multiplicative reasoning, meanings for slope, interpretations of graphs, and connections across multiple representations. We report in this article on the major differences related to multiplicative reasoning as evoked through the first task posed in the water-pumping context (the Pool Task, shown in Figure 3). Student reasoning for other tasks in the water-pumping situation are reported in Lobato and Rhodehamel (2009).

In the Pool Task, students were provided with a table showing the amount of water in the pool over time and were asked whether or not the water was being pumped equally fast over time. This task assesses how students reason informally about the attribute measured by the slope of the function (i.e., the "fastness" of pumping). The task is challenging because the values displayed for the independent variable do not increase by a constant amount.



Water is being pumped through a hose into a large swimming pool. The table shows the amount of water in the pool over time. The amount of water is measured in gallons. The time is measured in minutes.

Do you think the water is being pumped equally fast over time or is it being pumped faster at certain times? How do you know? How fast is the water being pumped into the swimming pool?

Time in minutes	Amount of water in gallons
0	0
3	6
5	10
9	18

Figure 3. The Pool Task.

Correspondingly, the classroom data were reduced to the set of lessons addressing the development of multiplicative reasoning—Lessons 1–6 in Class 1 and Lessons 1–5 in Class 2 (as shown in Table 1)—based on descriptive logs (Miles & Huberman, 1994) of what transpired in each lesson. Only data from the lessons addressing content directly related to the development of multiplicative reasoning with the two quantities involved in a linear function (as indicated by the shading in Table 1) are reported in this article.

Methods of Analysis

Analysis of interview data. We analyzed the interview data by inferring categories of reasoning, using what Miles and Huberman (1994) call a mixed approach in which some categories of reasoning were derived from the literature (e.g., additive reasoning) while other codes (e.g., reasoning with two uncoordinated recursive sequences) were induced using *open coding* from grounded theory (Strauss, 1987). Strauss and Corbin (1990) acknowledge the use of literature as legitimate within a grounded theory approach, as long as one is not overly constrained by existing categories.

Inferences were made on the basis of regularities in students' verbal utterances, gestures, and written inscriptions (following the multimodal approach of Arzarello, Paola, Robutti, & Sabena, 2009). Verbal utterances made during interviews were transcribed using Transana (Woods & Fassnacht, 2009) and synchronized with the video, allowing us to coordinate gestural and spoken evidence. Additionally, when students recorded their work on interview task sheets, these written records were coordinated with students' pointing, gesturing, and speaking.

We then identified connections between categories using axial coding (Strauss, 1987).

Table 1
Lesson Table Showing the Development of Linear Functions for Classes 1 and 2

Lesson	Class 1	Class 2
1	Explore ways to measure the "fastness" of travel.	Use small construction-paper squares to explore a visually growing pattern.
2	Investigate how changes in time and distance affect speed.	Use toothpicks to investigate a second visually growing pattern.
3	Use software to find time and distance values that produce the same speed.	Explore a third visually growing pattern. Make tables of data for all three patterns.
4	Explain why different distance and time pairs generate the same speed.	Discuss different relationships and patterns that students saw for the third pattern. Begin graphing the patterns.
5	Find and justify additional "same speed" values, including unit ratios.	Continue graphing. Write equations. Introduce slope.
6	Develop a verbal rule to express the relationship between distance, time, and speed.	Change m and b values in $y = mx + b$ using graphing software. Introduce stairs to discuss slope.
7	Graph a set of "same-speed" values. Write equations in $y = mx$ form.	Measure and compare the steepness of sets of stairs.
8	Make connections between tables, equations, and graphs.	Introduce the slope formula as rise/run. Find the slope of staircases and lines.
9	Introduce slope as a rate (e.g., speed).	Explore different stairs with the same slope using equivalent fractions. Find slopes of functions for the visual patterns.
10	Find the slope of a line. Informally explore the $y = mx + b$ case in a motion context.	Use staircases on graphs to find the slope of a line. Review.

Note. Only data from the lessons addressing content directly related to the development of multiplicative reasoning with the two quantities involved in a linear function (as indicated by the shading) are reported in this article.

This led to two supercategories, namely, (a) coordination of two quantities that preserves a multiplicative relationship and (b) nonmultiplicative reasoning. This allowed us to report group trends in reasoning on the Pool Task across the two classes.

Analysis of classroom data. We conducted four separate analyses of the reduced classroom data set. In the first analytic pass, we inferred centers of focus (CoFs) by using open coding from grounded theory, which involved breaking the data into manageable chunks and then identifying and naming categories into which the observed phenomena could be grouped (Hoepfl, 1997). To categorize the mathematical features that students noticed, we typically began by grouping similar words and phrases from students' verbal contributions (to both small-group work and whole-class discussions), and then coordinated this with what students pointed to when working with the computer, gestures made while presenting their work to the whole class, and written inscriptions from class activity sheets. Furthermore, our identification of multiplicative reasoning as a theme provided a source for what Strauss and Corbin (1990) call *theoretical sensitivity*—the "ability to give meaning to data" (p. 42)—which contributed to us being aware of data related to which quantities students noticed. For example, when a student in Lesson 1 of Class 2 remarked that "the pattern is growing by adding 3 unit squares each time," we were attuned to his noticing the number of squares in each figure and his lack of mention of a second quantity (which the teacher called the "step number" or ordinal placement of each figure in a visually growing pattern). The phrase "growing by adding 3" suggested the student's noticing of the change in the number of squares in each figure. Consequently, we tentatively labeled this center of focus *additive growth* and then scrutinized the student's remaining verbalizations, gestures, and written work during the lesson, both for additional evidence of this CoF, as well as disconfirming evidence (e.g., any mention of the quantity of step number).

Our process of inferring CoFs proceeded student by student, lesson by lesson, and class by class, using the constant comparative method of grounded theory (Cobb & Whitenack, 1996; Glaser & Strauss, 1967). For example, after determining a CoF for one student in a lesson, we would move to a second student in the same lesson to determine if the CoF was shared. If not, we defined a new CoF. Six CoFs emerged for the targeted lessons across both classes and are summarized in Table 2. In the Results section, we report the number of students who appeared to notice each CoF. The totals across CoFs for a particular lesson will not always equal the number of students in the class for two reasons: (a) one student may exhibit evidence for more than one CoF, or (b) there may be insufficient evidence to categorize a particular student (e.g., if the student did not share his thinking in class or on an activity sheet). Finally, we looked for evidence of the six identified CoFs in the lessons outside of the reduced set of lessons (i.e., in Lessons 7–10 in Class 1 and Lessons 6–10 in Class 2) to enable us to make claims regarding the stability of particular CoFs (but not to identify new CoFs). To answer the first research question (which is aimed at understanding the consequences of student noticing for subsequent reasoning), we then identified conceptual connections (in the spirit of Maxwell, 2004) between differences in the group trends of reasoning on an interview task across the two classes and differences in what students noticed mathematically during instruction (the centers of focus) across classes.

The next three analytic passes through the videotaped classroom data were conducted to inform our response to the second research question (which is aimed at explaining how particular centers of focus emerged or shifted). Specifically, we sought to offer an explanation of how CoF 3—a focus on composed units of two

Table 2
Inductive Codes Used to Identify Centers of Focus in the Classroom Videotaped Data

	Center of Focus (CoF) Codes	Description of Codes	Examples	Evidence
Class 1	**CoF 1:** Uncoordinated distance and time	To generate same-speed values, the student attends to both distance and time but uses guess-and-check to generate values rather than using a relationship between distance and time.	Guessing that 40 cm in 6 s will be the same speed as 20 cm in 8 s (possibly by doubling the 20 cm but not the 8 s).	Regularities in the values students enter into the computer and record on an activity sheet; reactions to the simulation; verbal accounts of reasoning.
	CoF 2: Coordinated patterns	To generate same-speed values, the student notices and uses patterns between distance and time but does not consistently attend to patterns that preserve the multiplicative relationship between distance and time.	Noticing that adding 10 cm and 4 s repeatedly produces same-speed values, but also attending to constant differences (i.e., subtracting 1 from 20 cm and 8 s to get 19 cm in 7 s).	Regularities in the values students enter into the computer and record on an activity sheet; reactions to the simulation; verbal accounts of reasoning.
	CoF 3: Composed unit of distance and time	The student appears to join distance and time together into a new unit, which is attended to as an entity that can be operated upon by iterating, partitioning, splitting, or combining these actions.	Explaining why traveling 50 cm in 20 s is the same speed as 10 cm in 4 s by attending to and repeating a "chunk" of "10 cm in 4 s" five times.	Verbal coupling of distance and time ("every 10 cm, it's using 4 s"); juxtaposing distance and time inscriptions; highlighting and pointing to "chunks" of distance and time in diagrams.
Class 2	**CoF 4:** Additive growth	The student attends to changes in the number of objects in consecutive figures in a visually growing pattern.	Building the next figure in a visual pattern by adding "what the pattern is growing by" to the previous figure.	Verbal reports ("it's growing by 3"); gestures (sweeping hand down list of numerals representing numbers of objects in each figure in a pattern) and written inscriptions (marking "+3" with an arrow between pairs of consecutive figures).
	CoF 5: Relationship between step number and number of objects	The student notices a relationship between the step number and the number of objects in the corresponding figure.	Predicting the number of objects in a given figure by multiplying the step number (which is also the number of objects in one "arm" of a figure) by the number of "arms" and then adding the number of objects in the "body" of the figure.	Verbal utterances in small groups; written reports of the relationship on activity sheets; circling associated parts of figures.
	CoF 6: A routine dominated by additive growth	The student attends to additive growth again, this time linking it to a routine for determining the number of objects in a targeted figure.	Finding the number of objects associated with Step 17 in the second pattern by multiplying the additive growth (4) by the step number (17) and adding the number of objects in Step 0 (3), but not coordinating the step number with the growth.	Verbal reports ("it's increasing by fours"); marking consecutive values in one column of a table with "4" and a bracket; written statements of the routine; verbalizations indicating that multiplying the step number by the growth was rule-based rather than understood in terms of the number of increases made in the pattern.

quantities—emerged in Class 1 and how students' center of focus shifted back to additive growth (CoF 6) in Class 2 because these were the two CoFs that appeared to be conceptually related to the trends in student reasoning during the interviews. In the second analytic pass, we examined students' and teachers' contributions to the discourse practices (focusing interactions) that occurred during the emergence of CoF 3 in Class 1 and the shift to CoF 6 in Class 2. We started with the discourse practices from Goodwin (1994)—highlighting and coding schemes—and then used open coding (Strauss, 1987) to identify additional discourse practices. Because the word *coding* has technical meaning in the educational research community, we chose instead to refer to this discourse practice as *renaming*. Three codes were used across the two classrooms to categorize the focusing interactions and are summarized in Table 3 (with details provided in the Results section).

In the third analytic pass, we investigated the features of mathematical tasks that appeared to be related to these targeted centers of focus. We first compiled all tasks posed to students during the lessons in the reduced data set. Then we identified the affordances and constraints of the mathematical tasks and activities, including representational features of problems that appeared to influence what students noticed mathematically (following Watson, 2004). An affordance (in the context of our study) is a relation between a resource in the environment (here a mathematical task) and the learners (Gibson, 1986). For example, mathematics tasks that utilize a variety of differently scaled graphs may afford students noticing the quantities on the axes better than tasks that only use scaling by ones. A constraint is a regularity that restricts activity in some way (Barwise & Perry, 1983). For example, a constraint built into a computer motion simulation may be that students can only enter distances and times (and not speeds). In turn, this constraint could influence what students notice when they watch the motion of several objects in the simulation. For example, having to enter time values may direct attention toward which object in the simulation finishes first.

In the fourth and final analytic pass, we ascertained the nature of the mathematical activity in each classroom by describing the rules for engagement in mathematical activity that appeared to govern students' and teachers' roles and that seemed to influence the number and nature of the candidates for centers of focus (following Cobb, Gresalfi, & Hodge, 2009; Cobb, McClain, & Gravemeijer, 2003). First, we watched each lesson in the reduced classroom data set and made a list of "general classroom obligations" for both students and teachers, inferred from the classroom dialogue and activity (Cobb et al., 2009, p. 52). For example, students may be obligated to share their own methods for solving problems, or they may be expected to listen to the teacher and take notes. Such lists were generated for each lesson because norms can shift over time (Lampert, 2001). Second, we selected the roles of students and teachers that appeared to be related to students' noticing. For example, if students are expected to explain their thinking in class and make sense of other students' methods, this could make a greater variety of mathematical information available as centers of focus. Finally, we created a narrative regarding the nature of mathematical activity (related to noticing) for each class.

Table 3
Codes Used to Identify Focusing Interactions in the Classroom Videotaped Data

Focusing Interaction Codes	Description of Codes	Examples	Evidence
Highlighting	Operating visibly on external phenomena, including the acts of labeling, marking, annotating, and gesturing.	*Student Example*. One student draws attention to a journey as being comprised of two "chunks" by using colored markers to add a line segment, tick marks, and the label "doubled" to another student's diagram.	Videotaped data of the student marking the diagram, her verbal explanation of what she was trying to accomplish, and her gestures.
		Teacher Example. The teacher draws attention to additive growth in the diagram of a visually growing pattern by drawing a curved arrow from the first to the second figure, writing "+4" above the arrow, and then repeating the inscription between the second and third figures in the pattern.	Videotaped data of the teacher marking the diagram, along with her accompanying verbal description of what she was trying to make salient ("we know the growth is 4").
Quantitative dialogue	Verbal communication that focuses attention on quantities as measureable attributes of objects.	A teacher presses a student to link numeric statements such as "twice as far" with the corresponding quantities in the student's diagram of a speed situation, and the student's responses shift from being numeric to more quantitative.	Questions from the teacher ("How can I see twice as far in your drawing?") with changes in student verbal responses and/or annotations (e.g., from speaking of numbers such as "10 and 20" to attending to and marking proportional distances in a diagram).
Renaming	Changing the name of a construct that has been previously defined, using a category of meaning from mathematical practice.	The teacher renames the "arms" in a visually growing pattern as the "growth."	Verbally changing one term to another term ("they use arms for growth here") and visually labeling one term with another term (writing "growth" beneath "# of arms").

Interview Results

In keeping with the methodological and conceptual design of the study, the interview results are presented first because they helped orient us to the analysis of the corresponding classroom data. Specifically, qualitative analysis of the interview data revealed distinct differences in how students reasoned with a table of linear data in a water-pumping situation (Figure 3). Seven of the eight students in Class 1 coordinated the two quantities in a way that preserved the multiplicative relationship between the quantities and correctly determined the pumping rate. In contrast, six of the nine students in Class 2 engaged in nonmultiplicative reasoning.

We illustrate these trends in reasoning by presenting the thinking of one student from each class. To determine that the water was pumped equally fast over time, Manuel, a student from Class 1, first found the difference between 3 min and 5 min as 2 min and the corresponding difference between 6 gal and 10 gal as 4 gal. His subsequent work is consistent with joining 2 min and 4 gal into a composed unit and partitioning that unit in half:

> So if you keep on going, it's just going to raise up 2 [moved index finger of left hand down from 3 min to 5 min in the time column, as shown in Figure 4a], and then right here, it's going to raise up 4 [moved pen in right hand down from 6 gal to 10 gal in the water column]. So it'd be 2 and 4. The minutes is 2 [moved left hand vertically back and forth between 3 min and 5 min in the time column], and this is going to be 4 gallons [moved left and right hands together up and down their respective columns]. So every minute, it's going 2 gallons, right? See, there's 4 in between here [motioned with pen in right hand between 6 gal and 10 gal] . . . so I split that in two. So it would be 1 minute, there's 2 gallons [wrote "1" to the right of the space between 3 min and 5 min and "2" to the right of the space between 6 gal and 10 gal, as shown in Figure 4a].

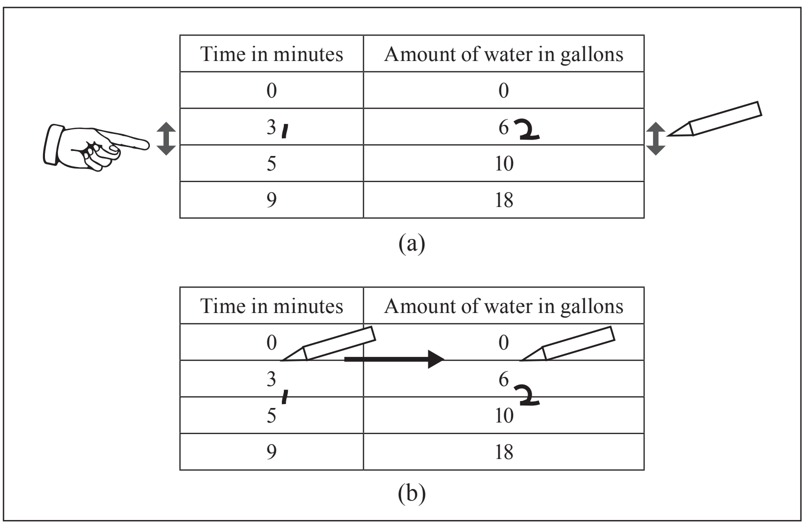


Figure 4. Authors' recreation of Manuel's gestures, along with his inscriptions, displayed during reasoning with the Pool Task.

Manuel's gesture with both hands to indicate the differences in time and water suggests a joining of the 2 min and 4 gal. His utterance "so I split that in two" suggests that he halved the 2:4 unit to create 1 min for 2 gal (also indicated in his inscriptions of "1" and "2"). Manuel then appeared to treat 1 min for 2 gal as a new unit, which he iterated repeatedly, suggesting that he saw new quantities in the table:

> They didn't show you 1 minute, but that would equal 2 [moved pen in right hand just below the zero in the time column and then horizontally swept across to just below the water column, as shown by the pens in Figure 4b]. And then 2 minutes, that would equal 4 [used similar horizontal sweeping gesture to that shown in Figure 4b]; and 3, 6 [horizontal sweeping gesture]; 5, 10 [horizontal sweeping gesture]; 4, 8 [horizontal sweeping gesture].

When asked how fast the water was being pumped, Manuel immediately responded, "Two gallons per minute." In a similar fashion, six of the remaining seven students in Class 1 appeared to compose time and water amounts into a unit and then iterated and partitioned that unit to conclude correctly that the pumping was equally fast over time and at a rate of 2 gal/min.

In contrast, six of the nine students from Class 2 engaged in nonmultiplicative reasoning. The most common strategy relied on univariate reasoning, as illustrated by Arcelia. She reasoned primarily with the gallon amounts, concluding that the water pumped the fastest where the difference in water amounts was the greatest:

Arcelia: Yeah, it's not, yeah, it's pumping like sometimes like much faster and sometimes slower but it's not equal.

Interviewer: What's not equal?

Arcelia: Oh, like it's not, like by the minutes, it's pumping sometimes more and sometimes less. Like right here they pumped 4 gallons from right here to right here [pointed to the 6 and then to the 10 in the gallons column] and then from right here to right here is 8 [pointed to the 10 and then to the 18 in the gallons column].

She went on to indicate, verbally and with gestures, that the pump was fastest between 10 gal and 18 gal, where the difference in the number of gallons is greatest. Other nonmultiplicative strategies included additive reasoning (e.g., one student subtracted 2 from both the 6 gal and 3 min to arrive at 4 gal in 1 min) and the creation of two uncoordinated recursive sequences (e.g., time values that "go by twos" and gallon amounts that have a "times 2" pattern).

Only three of the nine students in Class 2 preserved the multiplicative relationship and correctly solved the problem. These three students used multiplication to describe patterns between the time and water amounts rather than using the composed-unit reasoning found among students from Class 1. For example, one student explained: "In 1 minute it pumped 2 gallons, so 2 times 3 (is) 6, 2 times 6 is 12, and 2 times 9 is 18 . . . I multiplied by 2 because that's how many gallons are pumping away."

To account for these differing trends, namely for Class 1 students to coordinate two quantities multiplicatively and for Class 2 students to reason nonmultiplicatively, we next turn to the analysis of student noticing in the classroom. Specifically, we present students' centers of focus during the lessons devoted to the development of multiplicative reasoning. In the Discussion section, we explicitly address our conjecture that what students notice mathematically in class has consequences for their subsequent reasoning by arguing for conceptual connections between the differences in group trends on the interview task and the differences in centers of focus that emerged across the two classes.

Results of Classroom Analysis

Centers of Focus

In this section, we present evidence to support the claim that different CoFs emerged across the two classrooms. Specifically, we demonstrate a general trend that students in Class 1 first noticed two uncoordinated quantities, then noticed coordinated numeric patterns (that did not consistently preserve the multiplicative relationship), and finally shifted to noticing a composed unit of the two quantities. In Class 2, the general trend was for students to first notice changes in a single quantity (its additive growth), then to notice a relationship between two quantities, but soon to shift back to noticing additive growth. Details and individual differences from these trends are reported in the subsections below. The codes for the six CoFs that emerged inductively during analysis are summarized in Table 2.

Centers of focus in Class 1. Before providing evidence for any centers of focus, we briefly sketch the relevant class activities. Many of the lessons in Class 1 were devoted to the Same Speed Task and its extensions. In this task, students were asked to enter 10 cm and 4 s for one character (a rabbit), using a speed simulation environment programmed in The Geometer's Sketchpad® (Jackiw, 1995), and to determine as many different distance and time values as they could for another character (a turtle) so that the two characters traveled at the same speed (see Figure 5). Prior to this activity, students investigated how changes in time and then changes in distance affected how fast characters travel, and they acted out different motion scenarios to explore which quantities seemed to affect how fast one traveled (as summarized in Table 1).

Center of Focus 1: Uncoordinated distance and time. When the Same Speed Task was first presented to the whole class in Lesson 3, the teacher asked for a volunteer to suggest a distance and time. Ashlee suggested doubling the time and the distance (20 cm in 8 s). The teacher then polled the class to see who agreed with Ashlee's suggestion, but no one else thought the characters would walk equally fast. When the simulation was run, there was a resounding "Ohhh" of surprise as students saw that the characters traveled at the same speed. This suggests that most students did not enter the activity coordinating distance and time values.

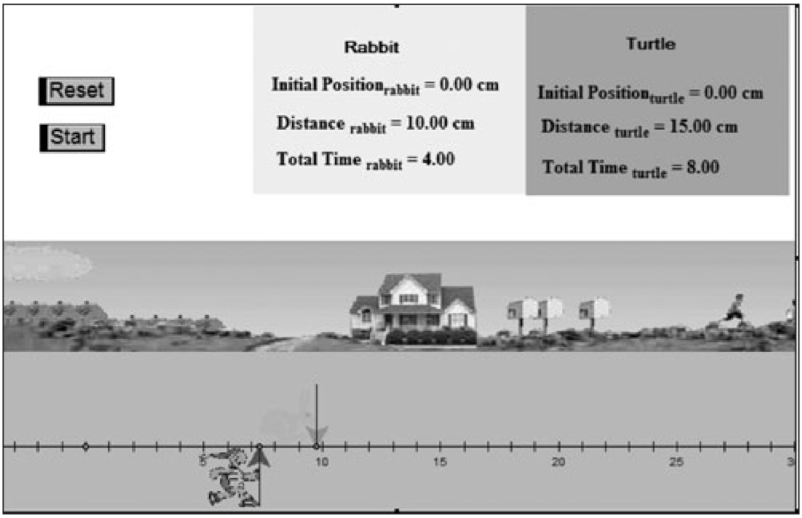


Figure 5. Screen capture from a speed simulation in Geometer's Sketchpad.

Students were then asked to work in small groups with the computer simulation to generate and test other "same-speed" values. Ashlee's contributions in her small group suggest that she was not coordinating distance and time beyond her one instance of doubling in the whole-class introduction to the activity. Specifically, she suggested to her group the following as same-speed values: 11 cm in 16 s, 22 cm in 16 s, and 11 cm in 5 s. Similarly, each individual student in Class 1 provided evidence of noticing both distance and time but in an uncoordinated manner. For example, Spencer and Hector started with 40 cm and 6s (see Figure 6). They ran the simulation and concluded that the turtle was faster as soon as it pulled ahead of the rabbit. For their second try, the students adjusted the time to 5 s and kept the distance as 40 cm, which made the turtle go even faster. When they finally changed the time in the correct direction, it was a large adjustment (40 s). Thus, they noticed both quantities but did not coordinate them. Additionally, during computer-simulation runs, they appeared to notice only the beginning of the race, just long enough to judge whether the characters were running "neck and neck," as indicated by pointing to the characters on the screen and then averting their eyes from the screen to record an evaluation of "faster," "slower," or "the same" on their activity sheets.

Center of Focus 2: Coordinated patterns. As the Same Speed activity progressed, there is evidence (from verbal reports captured by the small-group cameras and written evidence recorded on the activity sheets) that each individual began noticing coordinated numeric patterns among the distance and time values. For example, one group of three students determined that they could add 10 cm and 4 s repeatedly to produce the same-speed values of 20 cm in 8 s, 30 cm in 12 s, 40 cm in 16 s, and so on. One student described this pattern as "adding 10 and then adding 4." However, this group also engaged in additive reasoning, namely subtracting 1 from each of 20 cm and 8 s to arrive at 19 cm in 7 s. Although both patterns suggest a coordination of values, the students did not appear to be consistently noticing patterns that preserved the multiplicative relationship between the quantities.

Distance	Time	Did the turtle go faster than, slower than or the same speed as the rabbit?
20cm	8	SAME
40cm	6	FASTER
40 cm	5	FASTER
40cm	40	SLOWER

Figure 6. Spencer and Hector's table for the Same Speed Task.

Center of Focus 3: Composed unit of distance and time. An important shift happened when students appeared to join distance and time values into a composed unit, which could then be noticed and operated upon (e.g., by iterating, partitioning, splitting, or combining these actions). As sample evidence of CoF 3, we present an episode from Lesson 4. The class had just finished using drawings to explain why traveling 20 cm in 8 s is the same speed as 10 cm in 4 s. As a follow-up task, students were asked to determine how long the turtle should take to cover 50 cm if he travels at the same speed as the rabbit (who goes 10 cm in 4 s). David began by drawing a line segment to represent the rabbit's journey, which he labeled with 10 cm and 4 s; he then repeated this 10 cm in 4 s "chunk" four more times (see Figure 7). Written evidence for a composed unit is found in each distance inscription aligned directly below the corresponding time. Verbal evidence includes David's coupling of the 10 cm and 4 s in his explanation: "*Every 10 centimeters, it's getting ahead 4 seconds* [emphasis added], and as soon as you get to 50 centimeters, it takes up 20 seconds." The teacher called on Manuel for further elaboration. Manuel began similarly to David, stating that "every 10 centimeters, it's using 4 seconds." He elaborated where each group of 4 s came from in the journey, pointing at each chunk of the journey to obtain a total of 20 s:

> Okay. Um, every 10, every 10 centimeters, it's using 4 seconds, so it'd be 4, 4, 4, 4, 4 [pointed in succession to each "chunk" in the drawing, moving from left to right]. And then you add all those up, it equals 20, and every 10, every 10, it's going 4 again, so it'd be 50 centimeters in 20 seconds.

Although Manuel counted the time values first, he quickly attended to the 10 cm and 4 s together by stating that "every 10, it's going 4 again." David's "chunked" diagram and Manuel's pointing gestures to the "chunks" also suggest that they noticed a composed "10 cm in 4 s" unit.

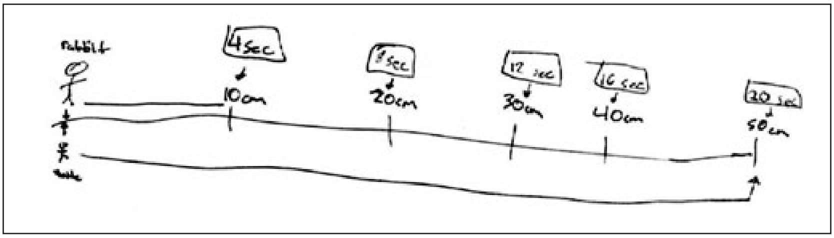


Figure 7. David's drawing used to show that it would take 20 s for the turtle to go 50 cm at the same speed as the rabbit.

CoF 3 is even more distinct when we look at another student, Spencer, who had not yet made the shift to CoF 3 during Lesson 4. For the task of explaining why walking 10 cm in 4 s was the same speed as traveling 20 cm in 8 s, Spencer created the diagram shown in Figure 8. The time and distance for each character are labeled in the diagram; however, the line representing 20 cm is not drawn as double the length of 10 cm and is not partitioned into two equal parts of 10 cm each. The latter part of the turtle's journey seems like an afterthought in Spencer's drawing, similar to students with CoF 1 only noticing the beginning part of the journey during computer-simulation runs. The proportional relationship between distances and times was also absent from Spencer's explanation, and he showed no evidence of iterating a "10 cm in 4 s" unit: "He [the rabbit] only has to go like 10 centimeters, but like he only has 4 seconds to do it, and so he'll have to rush, and the turtle has 8 seconds to do 20."

In summary, students with CoF 3 created "chunked" diagrams, juxtaposed distance and time labels in the drawings, verbally coupled distance and time, and often gestured to the "chunks"—all of which suggest that they noticed and used a composed unit of distance and time. There was evidence of this shift to CoF 3 for six students by the end of Lesson 4, and for the other two students in the class by

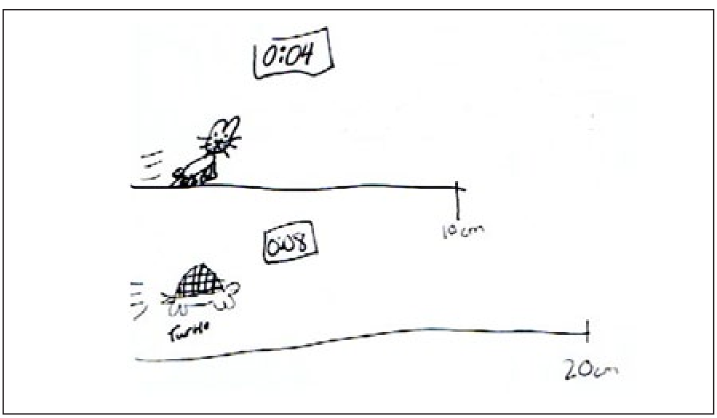


Figure 8. Spencer's drawing used to explain why traveling 20 cm in 8 s is the same speed as going 10 cm in 4 s.

the end of the instructional unit. Looking for evidence of CoF 3 outside the reduced data set revealed that once students noticed CoF 3, they did not shift back to CoF 1 or 2 (see Figure 9).

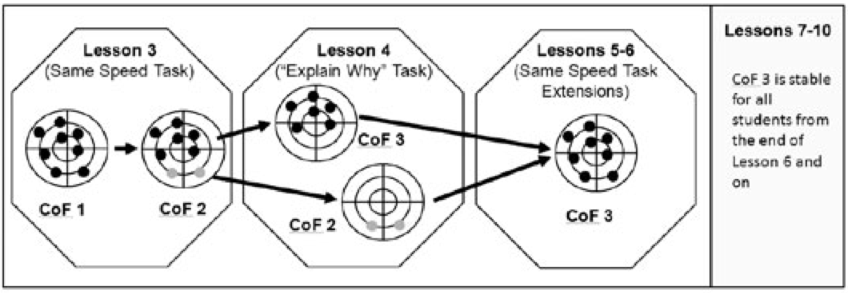


Figure 9. Summary of centers of focus for Class 1.

Centers of focus in Class 2. In Class 2, there were also shifts in the centers of focus. In this section, we provide evidence for three CoFs over a 5-day period (Lessons 1–5). During these lessons, students explored three visually growing patterns. Each figure in a pattern consisted of objects such as squares or toothpicks, and each consecutive figure had more objects than the previous figure. The ordinal placement of each figure was designated by step number; thus, the first figure in a pattern was called the figure in Step 1.

Center of Focus 4: Additive growth. During Lesson 1, students worked in small groups with manipulatives to find the number of squares associated with Steps 4, 5, and 10 in a visually growing pattern (see Figure 10). Students appeared to notice the change in the number of squares for figures in consecutive steps (without attending explicitly to the step number), as indicated by remarks such as, "Every time it is adding three more squares to form another row on the bottom." For example, Matt recorded "+ 3" with an arrow between pairs of consecutive figures on his activity sheet and wrote "I think it is growing by 3" (see Figure 10). Because the growth was not explicitly coordinated with the step number (as in "three squares per step number"), we call this center of focus *additive growth*. During Lesson 1, all nine individuals provided evidence of CoF 4 through their verbal contributions to small-group work or to the whole-class discussion, gestures, and/or written explanations and annotations.

Further evidence that the growth was not connected explicitly to the step number was found in Lesson 2. The class had been working with toothpicks on a second growing pattern (see Figure 11). Students noticed that "they're [the toothpicks] increasing by four," and that "you add four each time." After finding the number of toothpicks in Step 4 by adding 4 repeatedly, the teacher asked the students to think of a shortcut for repeated addition:

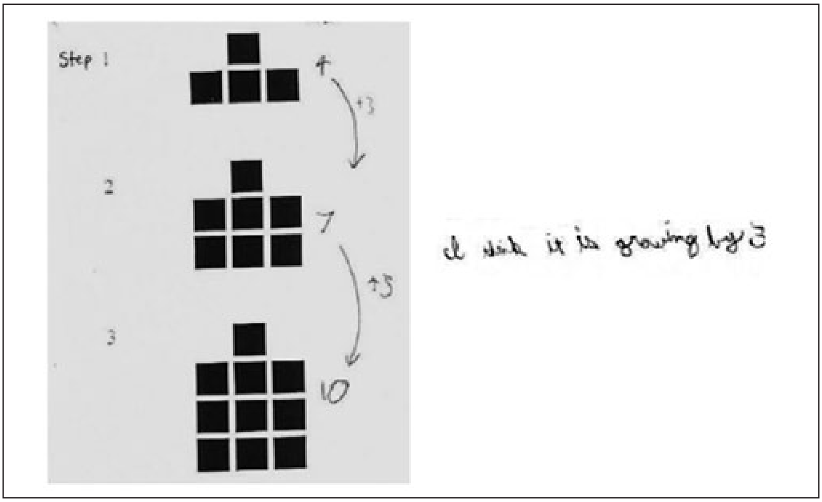


Figure 10. Matt's annotations and statement of the pattern he discovered for the first visually growing pattern used in Class 2.

> Now, do you notice anything if I keep adding 4 plus 4 plus 4 plus 4 plus 4, my hand's going to get tired? Is there another way to make this a little easier for me to write maybe? . . . Is there a shortcut?

Six students used arithmetic operations that were not connected to the step number to identify different number patterns. For instance, one student tried exponents ("4 to the power of 4"), and another used complicated addition patterns ("4 plus 4, plus 3, plus 2, plus 1, plus 3"). Three students did use multiplication (multiplying the step number by the growth, here 6 × 4, for Step 6) but then added the number of objects in Step 1 rather than Step 0, suggesting that the step number was not coordinated with the number of times the "growth" was being added.

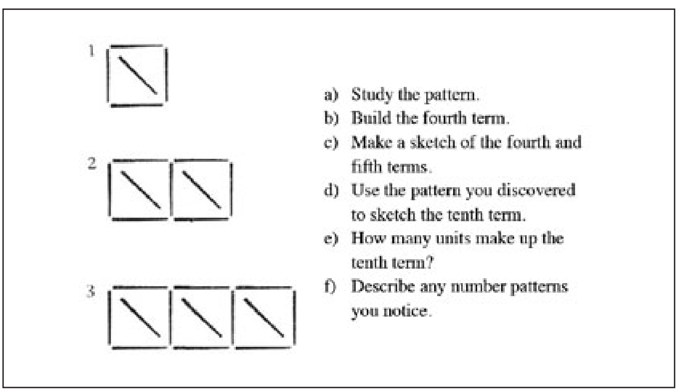


Figure 11. The second visually growing pattern and accompanying task used in Class 2.

Center of Focus 5: Relationship between step number and number of objects. During Lesson 3, six of the nine students appeared to shift from noticing additive growth (CoF 4) to noticing a relationship between the step number and the number of objects in the corresponding figure (CoF 5), while CoF 4 remained in play for the other three students during this lesson. Consequently, the step number became more explicit in students' talk, and it was no longer necessary to "build up" a pattern from the beginning by repeatedly adding the "growth" to determine the number of objects in a given figure.

This shift to CoF 5 for the majority of the students occurred while they worked in small groups with a third visually growing pattern (see Figure 12). For example, Domingo noticed that "the number of the step is going to be the number [of] squares there are in the arms" of a figure for a given step in the pattern (see Figure 13). For the figure associated with Step 3, Domingo circled the step number (3) as well as three squares in one arm of the associated figure. Using this relationship, Domingo described how to find the total number of squares for a given step number: "Multiple [sic] the number of the step by the number of arms, then add the 3 middle ones." Similarly for Step 10, he noted that there are "10 squares in the arm in Step 10," so he multiplied 10 by 4 and then added "the 3 middle ones" to arrive at 43 objects in the figure.

The explicit noticing of the step number and its coordination with the number of objects by the six students who shifted to CoF 5 is promising mathematically because this relationship can be developed into a multiplicative relationship. Specifically, as one begins to notice that there are 4 times as many objects altogether in the "arms" as the corresponding step number, then 4 could become the ratio of the number of objects in the arms to the step number.

Center of Focus 6: A routine dominated by additive growth. During Lesson 4, the teacher funneled the class toward adopting a general routine for finding the number of objects in a figure by multiplying the step number by the growth and then adding the number of objects in Step 0. We provide evidence that this routine was dominated by additive growth and that the step number appeared to be only

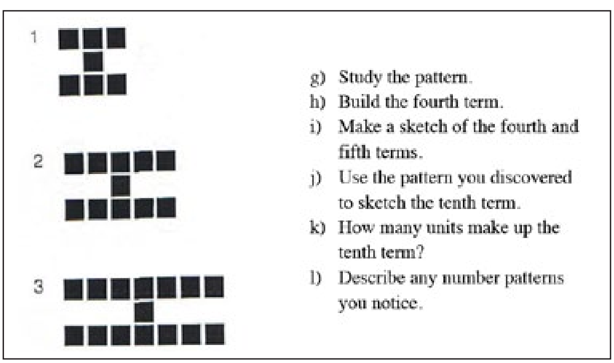


Figure 12. The third visually growing pattern and accompanying task used in Class 2.

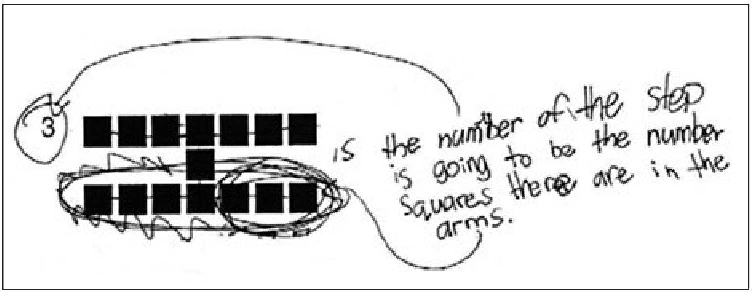


Figure 13. Domingo's statement of the relationship he noticed in the visually growing pattern.

loosely connected to the growth for the students. However, evidence for students' centers of focus was more difficult to establish in Lesson 4 because the nature of the mathematical activity became more teacher directed; thus, student responses were less elaborated. Lesson 4 began with the teacher validating CoF 5 (noticing a relationship between the step number and the number of objects in each arm). However, in a discursive interaction, the "arms" were renamed the "growth," which we claim directed attention back to additive growth. Our purpose in this section is to provide evidence that each of the students noticed and used the growth-dominated routine by the end of Lesson 5. The teacher-directed nature of the lesson and the teacher's use of renaming will be examined in detail in The Nature of Mathematical Activity and Focusing Interactions sections, respectively.

First, the most compelling evidence of a shift away from CoF 5 is the fact that the relationship is only briefly mentioned again once (by two students) during the unit (in Lesson 5), despite its prominence in Lesson 3. Specifically, one student said that he remembered the method from a previous lesson. However, when a second student stated the method, he did so not in terms of a relationship between the step number and the number of objects in a figure but rather in terms of additive growth. He described the method as multiplying the step number by "what it keeps going by" (the additive growth), plus "the middle ones."

Second, many students appeared to appropriate the focus on additive growth that was present in the way the tasks were posed in Lessons 4 and 5. For example, in Lesson 4, the class revisited the toothpick pattern (Figure 11), and students were asked to determine the number of toothpicks for Step 17 (emphases added):

Teacher: Look at the function and see how many toothpicks it's *growing by*. Okay, let's talk about the growth. Here's 5 [referring to the number of toothpicks associated with Step 1], and you said it was *growing* . . .

Soledad: *By fours*.

Teacher: Okay, by fours. That was the toothpicks we added to the end, right? If it *grows by 4*, excuse me, that shouldn't be a 4. What should it be [referring to the number of toothpicks associated with Step 2]?

Chanise: 9.

Teacher: So this is the *growing by the 4*. Now, what if I add 4 more toothpicks?

Students: 13.
Teacher: Go ahead and see if you and your group can figure out what Step 17 has.

The students' responses suggest that they noticed the "growing by 4" pattern. Similarly in Lesson 5 and for the first visually growing pattern (see Figure 10), the teacher asked the class for the values at Steps 1, 2, 3, 4, and 5 in succession. Students responded with 4, 7, 10, 13, and 16, respectively, thus, drawing attention to an increase by 3.

Third, when students determined the number of objects associated with Step 17 for the first and second patterns during these two lessons, they had difficulty adjusting the product of the step number times the growth, suggesting that the step number was not coordinated with the number of times the growth was "added." For example, to find the number of toothpicks in the second pattern, Bahir said it should be "17 × 4, because it's increasing by fours." When the teacher checked in with Bahir's group, she prompted them to make an adjustment of 1 to their incorrect answer of 68 by asking how many toothpicks are in Step 0. Although Bahir could correctly determine that there was one toothpick in the figure associated with Step 0, he said that it was already included in the 68. This suggests that Bahir had not coordinated the 17 from Step 17 with its meaning as 17 groups of four (from Step 0 to Step 17), which does not include the one toothpick found in the figure associated with Step 0 but rather would need to be added to it to obtain the total number of toothpicks in the figure associated with Step 17.

Fourth, when the routine "step number times growth plus Step 0" was presented in the whole-class discussion, "growth" was explicitly emphasized as being additive rather than multiplicative. For example, in Lesson 5, Soledad described the routine for finding the number of squares in Step 17 for the first pattern: "I multiplied 17 times 3 . . . then that's 51 . . . and then I added 1 on top." When the teacher asked the class where the 3 came from, Matt responded with attention to additive growth: "The 3 is the 3 that you are always adding on." The teacher solidified the routine by substituting *n* (which she called "any number") for the 17 in Soledad's method, writing "3* *n* + 1," and labeling the 3 as the "growth," the *n* as the "Step #" and the 1 as "Step 0" (see Figure 14). In each of these instances, the routine appeared to be a series of computations, and there is no evidence that students attended to a relationship between the step number and the growth, although it is possible that some students did.

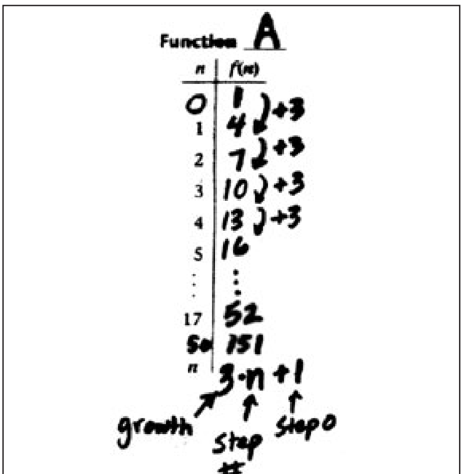


Figure 14. The Class 2 teacher's annotation of the routine for finding the number of objects in a figure associated with Step *n*.

By the end of Lesson 5, each of the individual students in Class 2 had provided verbal or gestural evidence of one or more of the four types of evidence for CoF 6 discussed above. Looking for evidence of CoF 6 outside the reduced data set revealed that CoF 6 was stable during Lessons 6–10 (see Figure 15).

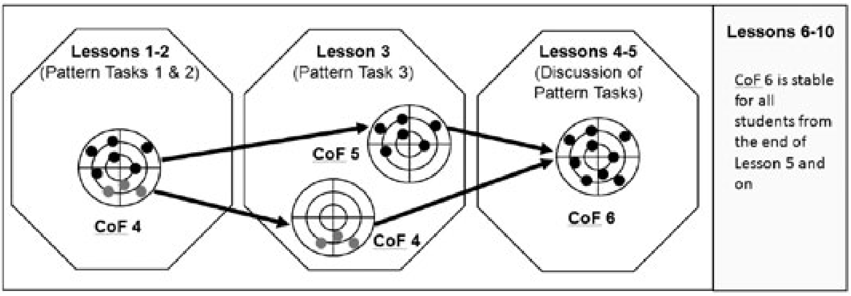


Figure 15. Summary of centers of focus for Class 2

Focusing Interactions

In this section we provide a partial explanation for the emergence of or shift in particular centers of focus by identifying *focusing interactions*—the discourse practices that appear to direct students' attention toward certain mathematical features. Specifically, we offer an explanation of how CoF 3—a focus on composed units of distance and time—emerged in Class 1 and how students' center of focus shifted back to additive growth (CoF 6) in Class 2. Although some new classroom episodes are presented as evidence, we also return to some classroom episodes reported in the CoF section, but with additional elaboration and using a different unit of analysis, namely the communication between teacher and students and the communication among students. Although many attention-focusing moves can be attributed to teachers, we also investigate the role students can play in shaping the emergence of centers of focus during student–student communication and in discursive interchanges with the teacher.

Focusing interactions in Class 1. We examine the role of two discourse practices in Class 1: highlighting and quantitative dialogue. The first derives from Goodwin (1994), and the second emerged as a result of grounded analysis (see Table 3 for a summary). *Highlighting* entails visible operations upon external phenomena, including labeling, marking, annotating, and gesturing. *Quantitative dialogue* refers to communication that directs attention to *quantities*, where a quantity (following Thompson, 1994) is a measurable attribute of an object, event, or situation (e.g., distance, density, or how fast an object travels). Quantitative reasoning is distinct from numeric reasoning because quantities can be imagined without numeric values and because numeric reasoning can occur without links to measurable attributes. Consequently, in a quantitative dialogue, a teacher may press a student to link numeric statements to the corresponding quantities in a given context, and the student's responses may shift from numeric to quantitative.

Highlighting. In the following episode, we explicate how a student's actions of highlighting provided support for a shift in focus to a composed unit of distance and time for other students during a whole-class discussion. As mentioned previously, students were asked to use drawings in Lesson 4 to explain why two distance–time pairs (10 cm in 4 s and 20 cm in 8 s) represent the same speed, and the first diagram presented in class (Spencer's) lacked a proportional representation of the turtle's distance compared to the rabbit's (see Figures 8 and Figure 16a). When Ana was called on next, she annotated Spencer's diagram by extending the line for the rabbit's journey so that it ended at the same distance as the turtle's journey, and she drew 10 tick marks, apparently to show that another 10 cm had been covered (see Figure 16b). She wrote "doubled" above this second "chunk" of the journey and explained:

> Then I put that this [pointed to the section she added] is the same length as this in between here and then there [swept her hand from left to right over the distance that Spencer had drawn for the rabbit], so it's just like the same thing . . . from here to here is 10 [pointed to the first 10 cm], and it's adding 10 more [pointed to her added section], and that'd be 20, and then it's the same length.

She then explained how doubling the time could also be seen in the drawing. Pointing to the 4 seconds labeled for the rabbit, Ana stated, "And then you add 4 more to that because you have to double it, then it'd be 8 seconds, so they're going the same speed."

This episode is significant because Ana's highlighting made salient the portion of the turtle's journey that occurred after the rabbit stopped. Prior to this event, many students had attended to the beginning of the journey to make claims regarding which character was faster (by noting who pulled ahead in the computer simulation). Ana made visible that the two units of 10 cm in 4 s are contained in the 20 cm in 8 s journey—one unit occurring when both characters were in motion, and the other after the rabbit had stopped. This may have allowed other students to see that the turtle's journey could be thought of as the rabbit completing his journey twice, and thus conclude that the characters traveled at the same speed.

Quantitative dialogue. The following episode illustrates how a quantitative dialogue between the teacher and a student appeared to make measurable attributes of the speed situation more salient for the student. This event occurred as students worked in small groups prior to the whole-group sharing described above. Ashlee had created the drawing shown in Figure 17a to explain why the rabbit (traveling 10 cm in 4 s) and the turtle (20 cm in 8 s) were going the same speed. The teacher asked Ashlee to explain her drawing:

Ashlee: Well, he's (the turtle) saying, I have more time but I have to go twice as far.

Teacher: How can I see twice as far in your drawing?

Ashlee: 10 and 20.

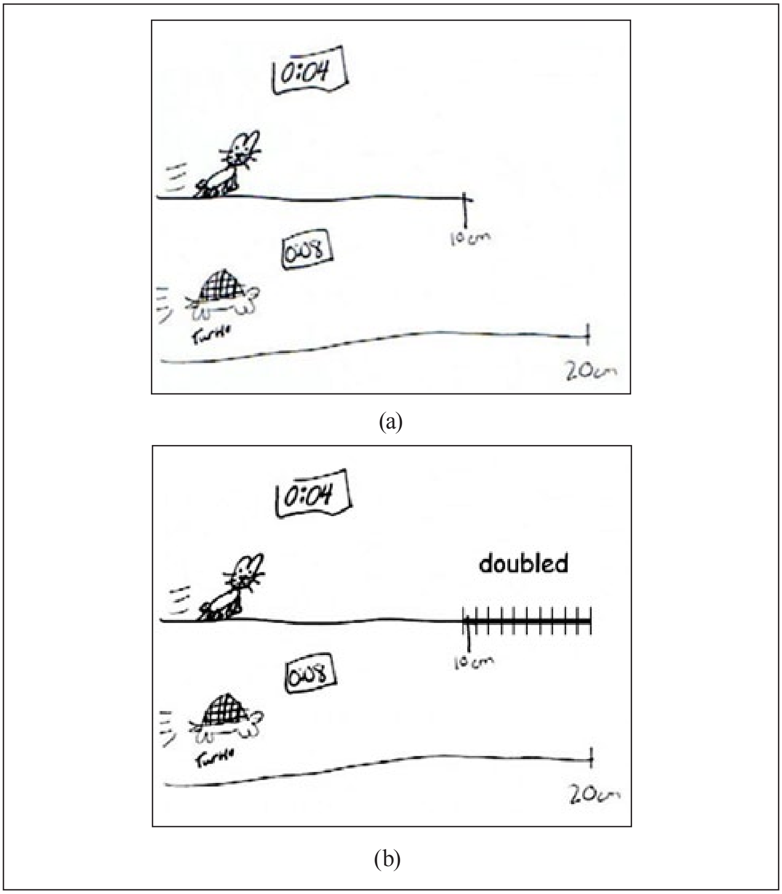


Figure 16. (a) Spencer's drawing used to explain why traveling 20 cm in 8 s is the same speed as going 10 cm in 4 s, and (b) the authors' recreation of Ana's annotations on Spencer's drawing.

Teacher: Is 10 a point or a distance?

Ashlee: A point? A distance. . . . I don't know!

Ashlee began by making the numeric statement "twice as far," which could have indicated a numeric calculation of $20 = 2 \times 10$. In response, the teacher appeared to request that Ashlee connect the numeric statement to what it signifies in the context, by asking "How can I see twice as far in your drawing?" Ashlee responded with another numeric statement, namely "10 and 20," as opposed to talking about measurable attributes of the situation, such as time or distance. The teacher persisted by asking Ashlee if 10 is a point or a distance, a quantitative question, which apparently puzzled Ashlee. The teacher then scaffolded attention to the 10 as a distance by asking where the character started:

Teacher: Where's the 0? Where do they start?

Ashlee: Oh! [drew a point at the far right of the horizontal line associated with the rabbit and labeled it with a zero, as shown in Figure 17b]

Teacher: Okay, so this is 10 centimeters? [gestured with thumb and index finger spread apart, spanning the distance between Ashlee's dot and the rabbit]

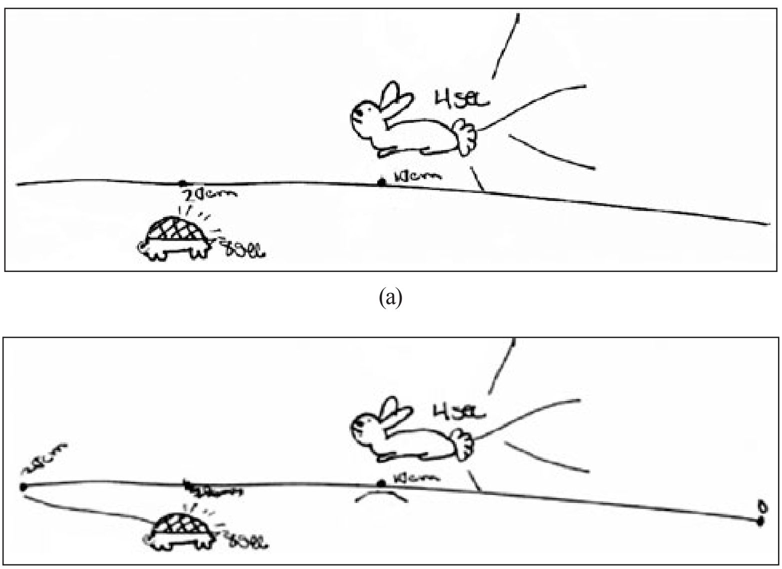
Ashlee: Yeah.

Teacher: How much is this between here and here? [gestured with thumb and index finger spread apart, spanning the distance between the turtle and the rabbit]

Ashlee: Not far enough! [drew a dot to the left of the line, wrote "20 cm" near the dot, and then connected the turtle with the dot, as shown in Figure 17b]

Once the teacher highlighted the 10 cm as the quantity of distance traveled by the rabbit, Ashlee appeared to notice that the distance she had drawn between the rabbit and turtle was not far enough. Ashlee then changed the turtle's distance so it was the same as the first chunk of the journey traveled by both characters, creating a new diagram in which the turtle visually appeared to have traveled twice as far as the rabbit.

This episode is significant because it shows how quantitative dialogue and highlighting functioned together to support a transformation of Ashlee's drawing. What first appeared to be a "comic strip" of the race, with locations and times identified, became a mathematical representation, where the relationship between quantities



(a)

(b)

Figure 17. Ashlee's (a) initial and (b) revised drawings used to explain why 10 cm in 4 s and 20 cm in 8 s represent the same speed.

was salient in that it was possible to see what was signified by “twice as far.” This quantitative dialogue also appeared to influence the next diagram that Ashlee created to explain why 30 cm in 12 s and 10 cm in 4 s represent the same speed (see Figure 18), because the turtle’s journey of 30 cm in 12 s is more clearly separated into three chunks of 10 cm in 4 s.

Focusing interactions in Class 2: Renaming. In this section, we offer a partial explanation for how the center of focus in Class 2 shifted back to additive growth by examining the role of *renaming*, a discursive move that is derived from Goodwin (1994). Renaming refers to changing the name of a previously defined construct, using a category of meaning from mathematical practice to classify and label some mathematical characteristic or property (summarized in Table 3). The simple act of changing a name may carry significant implications regarding how one perceives the given entity as well as the relationships associated with it.

At the beginning of Lesson 4, there was a particularly influential episode in which renaming, together with highlighting, appeared to support a shift in focus back to additive growth. Initially, the teacher validated the relationship that students noticed between the step number in a visual pattern and the number of objects in the associated figure (CoF 5). Specifically, the teacher displayed the pattern (see Figure 19), highlighted the number of squares in one arm in the figure associated with Step 1, wrote “1” below the figure, circled the four arms, and wrote “• 4” to the right of the 1. She continued in a similar manner, highlighting the next two figures in the visual pattern. When she got to the statement “3 • 4 + 3 = 15,” she labeled the first 3 as the *step number*, circled the number of squares in one arm, labeled the 4 as the *number of arms*, and labeled the second 3 as the *middle squares*.

Then one student, Bahir, appeared to focus attention back on additive growth by asking, “If you want to go from Step 2 to Step 3, since we know it’s going by 4, couldn’t you just do 11 + 4”? This publicly voiced observation initiated an act of

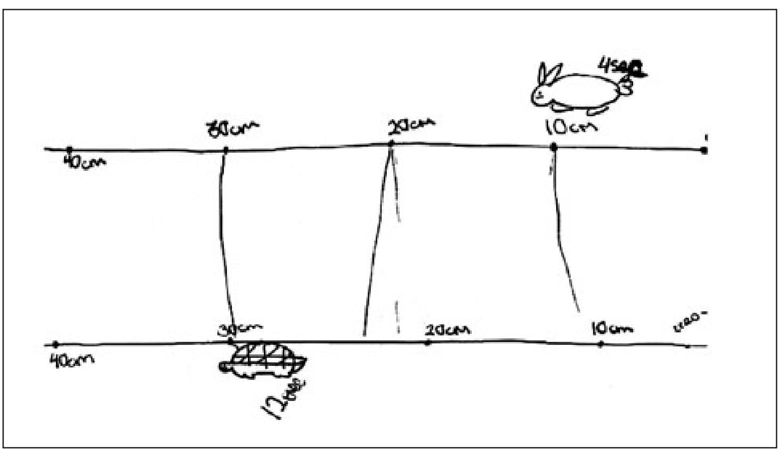


Figure 18. Ashlee’s drawing used to explain why traveling 30 cm in 12 s is the same speed as going 10 cm in 4 s.

renaming by the teacher, in which the relationship between the step number and number of squares in a figure became associated with additive growth. Specifically, the teacher validated Bahir's idea and highlighted the additive growth of 4 in the diagram with arrows and "+ 4," as shown in Figure 20. In a crucial move, the teacher renamed the number of arms as the growth and wrote "growth" beneath "# of arms" (also shown in Figure 20). In her words:

> Yes, good. We know the growth is 4 . . . [annotated with arrows and "+ 4"] . . . So the growth, if I want to know which number talks about the growth, now this person used what for growth? What word? It's growing by 4. Okay, arms. *So they use arms for growth here*. . . . Every time it's growing by 4. That's what we need to focus on, the growth [emphasis added].

The renaming of "number of arms" as "growth" is particularly interesting because the 4 representing the number of arms does not change and is not the same as additive growth, despite the fact that the numerical values are the same. By renaming "number of arms" as "growth," the teacher invoked the network of meanings that had come to be associated with growth in this classroom. In fact, "growth," with its synonyms of "increase" and "going by," was referenced verbally no less than 172 times in the

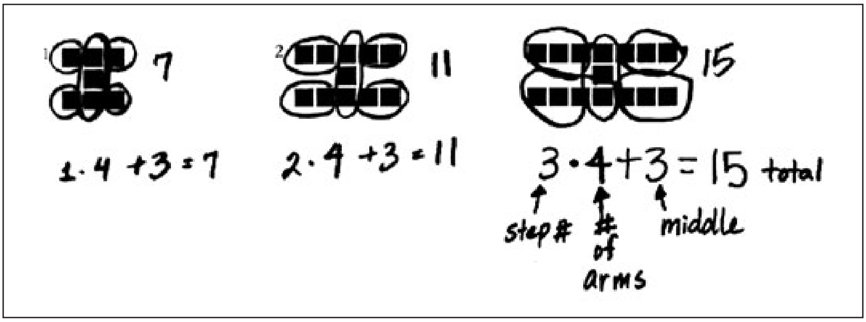


Figure 19. The Class 2 teacher's annotations highlight the relationship that students discovered between the step number and the number of objects in the associated figure in a visual pattern.

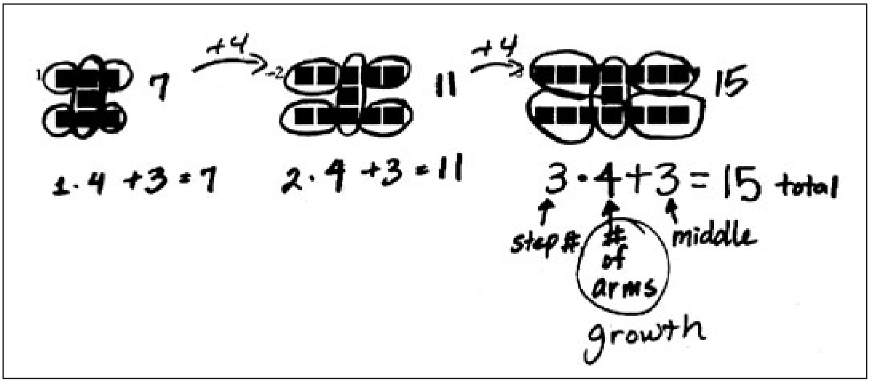


Figure 20. The Class 2 teacher renames the "# of arms" as the "growth."

first few lessons. As a category of meaning in Class 2, growth referred to the change in the number of objects associated with consecutive step numbers, where the step number was treated as an implicit quantity. In the discursive exchange articulated above, the renaming act was followed by an explicit verbal command to the students: "That's what we need to focus on, the growth." Indeed our analysis indicates that this was a turning point with respect to what students noticed. As presented previously, CoF 5 was mentioned only once during the remainder of the unit (by 2 students) despite its prominence in Lesson 3, and additive growth dominated the routine that emerged as CoF 6 ("step number times growth plus Step 0").

Features of Mathematical Tasks

In this section we present the results of our investigation regarding how the features of mathematical tasks may have afforded and constrained what students noticed. Although Goodwin (1994) treated the role of the task implicitly as a context for the interaction under investigation, we offer an explicit (albeit brief) analysis of the features of the tasks used in each class. Specifically, we present two major differences in the nature of the tasks, which appear to be connected to the differences in the centers of focus that emerged across the classes.

First, the same speed tasks used in Class 1 seemed to focus attention on both quantities of time and distance. Furthermore, by asking students to discover same-speed values, attention was also drawn to an emergent quantity, speed, which can be conceived as a ratio of distance to time (and is the attribute measured by the slope of the function). In contrast, the visual patterning tasks used in Class 2 seemed to make the independent variable (step number) implicit, which likely inhibited the formation of a multiplicative relationship between the quantities of step number and number of objects in the associated figure. It was not necessary to coordinate these quantities to correctly respond to the patterning tasks. As seen in the analyses of CoFs 4 and 6, students could simply use addition recursively to arrive at correct answers without forming an explicit relationship between the step number and the number of objects, or they could use multiplication without understanding the conceptual relationship between the step number and the number of groups of "growth" that were needed.

Second, the continuous nature of distance and time in the speed context in Class 1 afforded more opportunities for ratio reasoning than the discrete nature of the patterning context used in Class 2. For example, being asked, "How many objects are in the figure for Step 2.5?" is not sensible in the patterning context. In sum, these two differences in task attributes—the explicitness of the independent variable and the use of continuous versus discrete quantities—appeared to contribute to the emergence of different centers of focus in the two classes.

The Nature of Mathematical Activity

In this section, we present the results of the fourth and final analytic pass through the classroom videotaped data, which explored the more global character of the nature of the mathematical activity in each class and how it may have been related to the centers of focus that emerged. We identify similarities and differences in the classroom cultures of the two classes in terms of the roles regulating who was allowed to talk and what types of contributions they could make.

The participatory organization that established the roles and expectations for students and teachers was similar during Lessons 1–3 but shifted during Lesson 4 in both classes, which coincided with a shift in the centers of focus in each classroom. Specifically, at the beginning of both instructional units, students were expected to work in small groups, generate and share their own methods for solving problems, and make sense of other students' methods. Both teachers moderated discussions, invited students to share their methods, and compiled student ideas.

During Lesson 4 in Class 1, the students were first expected to explain why the patterns that they discovered when working with the computer simulation held and to create drawings as tools for reasoning. The novelty of this new role resulted in protests from students. The teacher did not give in to the complaints, instead upholding students' responsibility to generate satisfactory explanations. The teacher prompted students to represent quantitative relationships from the speed context in their drawings and, as a result, helped establish a productive sociomathematical norm (Cobb & Yackel, 1996), namely that a satisfactory justification should involve relevant quantities and quantitative relationships. This appeared to contribute to the emergence of a composed unit as the center of focus during Lesson 4.

In Class 2, the nature of the mathematical activity also changed in Lesson 4, coinciding with the shift in centers of focus back to additive growth. The nature of the mathematical activity changed from being student centered to teacher centered—a change that lasted until the end of the unit. Specifically, there was less work in small groups, instruction became more tightly guided, and students often responded to questions with short answers. This change in the nature of the mathematical activity appeared to contribute to the effect of funneling attention toward additive growth. When students are expected to create and share their own methods, it seems likely that the number of potential centers of focus will increase as a result of the proliferation of different ideas from students. On the other hand, if the teacher dominates the classroom interaction, then there are fewer opportunities for students' ideas to be voiced, leading to a narrowing of potential foci available to the class as a whole. This is not to say that there may not be students who notice different mathematical features than those promoted by the teacher, only that these foci are unlikely to proliferate without a public means of communication.

Discussion

In this study, we examined students' mathematical noticing by using our four-component focusing framework to analyze two middle grades mathematics classes that shared the same overarching content goals. An analysis of the centers of focus revealed significant differences in what students from each class noticed mathematically. Specifically, there was a general trend for students in Class 1 to first notice two uncoordinated quantities, then notice coordinated additive number patterns, and finally notice coordinated quantities that preserved a multiplicative relationship

(CoF 3)—a center of focus that was eventually shared by all students in the class and that remained stable until the end of the unit. In Class 2, the general trend was for students to first notice changes in a single quantity (its additive growth), then notice a relationship between two quantities, but soon shift back to noticing additive growth (CoF 6)—also a convergent and stable center of focus.

We then demonstrated that differences in the focusing interactions, features of mathematical tasks, and nature of mathematical activity in each class worked together to bring about the divergent centers of focus across classes (i.e., CoF 3 versus CoF 6). Fine-grained discursive moves, such as the ways students and teachers in Class 1 annotated diagrams for each other and the teacher's press for students to talk in terms of measurable attributes of situations, helped provide opportunities for students to attend to two quantities in a coordinated manner. These discourse practices occurred while students engaged with tasks that involved two explicit continuous quantities and in mathematical activity that emphasized explaining. In Class 2, the renaming of "number of arms" as "growth" was pivotal in drawing students' attention back to additive growth. This discourse practice occurred while students engaged with tasks that emphasized one discrete quantity over the other and in teacher-centered mathematical activity (from Lesson 4 on), which inhibited a focus on the relationship between quantities that students had noticed in an earlier lesson.

Ramifications of Student Noticing

Our examination of the relationships between students' centers of focus during class and how they reasoned in post-instructional interviews led us to the following conclusion: what students notice mathematically has consequences for their subsequent reasoning. Specifically, the differences in the centers of focus in each class appear to be conceptually connected to differences in group trends across classes regarding how students reasoned on an interview task set in a novel context and with nonconstant intervals for the independent variable. The convergent center of focus in Class 1 (CoF 3) involved attention to distance and time as quantities that were coordinated in a manner that preserved the multiplicative relationship (in this case as a joined or composed unit). Correspondingly, all but one student in the interview coordinated the time and water amounts in the interview task in such a way that allowed them to conclude correctly that the water pumped at a constant rate of 2 gallons/minute. The convergent center of focus in Class 2 (CoF 6) involved attention to the additive growth of the number of objects between consecutive figures in visually growing patterns (without coordination with the second quantity—step number). Correspondingly, each of the nonmultiplicative reasoning strategies used during the interview by six Class 2 students involved differences between consecutive values of a single quantity, which is at the heart of the center of focus on additive growth in Class 2.

Our choice to present conceptual connections between group trends across the interviews and the centers of focus is related to our argument regarding the ramifications of noticing. We posit that what students notice mathematically becomes a basis from which they generalize their learning experiences to reasoning in subsequent situations. A variety of different ways of reasoning in a later situation may be possible, but each is grounded in what was noticed. For example, the six students in Class 2 who reasoned nonmultiplicatively during the interview used three different strategies—additive reasoning, reasoning with two recursive and uncoordinated sequences, and univariate reasoning—yet each strategy involved additive growth. Thus, identifying centers of focus may not permit an accurate prediction of fine-grained strategy use, but it can inform general expectations regarding the consequences of noticing for reasoning on novel tasks.

Implications for Teaching and Research

This work advances the field of mathematics education along several dimensions. First, we consider the implications of this research for teachers and teaching practices. This study can help make teachers aware that for any mathematical topic there are multiple mathematical features that their students may or may not notice, some of which form the foundation for more productive mathematical ideas than others. If students are noticing other mathematical features, it may be impossible for them to form the intended mathematical idea. For example, the research reported in this article can be generalized to suggest that if teachers want students to understand covariation (for any type of function—linear, quadratic, exponential, and so on), then students need to notice two quantities, not just one. Additional examples can be drawn by interpreting other mathematics education literature from a noticing perspective. Specifically, if teachers want their students to reason quantitatively (in the sense of Smith & Thompson, 2008), then the students need to notice measureable attributes of objects and situations, not just numbers and numeric patterns. If teachers want their students to form some meaning for division, either partitive or quotative, then the students need to notice "groups of" discrete objects or groupings within continuous entities.

This research also indicates that teachers can play an important role in directing students' attention toward or (unintentionally) away from what is centrally important for students to notice for a given topic. Consequently, teachers should be aware of the importance of subtle and fine-grained discursive moves, such as renaming or highlighting particular information on the board (via pointing or annotating). For example, Teacher 2 in this study started the unit with problem-solving activities involving visual patterns that students worked on collaboratively with manipulatives (toothpicks and small squares). The teacher listened to, and in fact displayed, the relationship that many of the students noticed between two quantities (namely between the step number and the number of objects in the associated figure). However, the teacher's renaming of "number of arms" as "growth" was part of a pivotal event that marked the shift of students' attention back to changes in a single quantity.

It is challenging for a teacher to be aware of the influence of such a subtle move during teaching, especially considering that we as researchers did not understand the importance of the renaming event until we retrospectively analyzed the data. Thus, teachers need a way to assess what students are noticing mathematically during class, both before and after particular instructional moves. In a subsequent study, one of the authors created a promising approach for gathering such information(Hohensee, 2011). Specifically, he developed a *noticing practice* by establishing a routine in which a student would display a diagram that he or she had constructed, while the teacher would call on several students to describe what they noticed in the diagram before the presenter explained his thinking. As a result, the teacher was able to assess which mathematical features were salient for students, from among a range of possible features, and determine whether or not what the students were noticing might be productive for their development of understanding of the topic.

This work also advances research in mathematics education. Its significance lies not simply in the offering of a local explanation for the emergence of differences in what students noticed in two particular mathematics classrooms and the ramifications of those differences on later reasoning, but also in the promise of the focusing framework as a general conceptual and methodological tool that can be used to investigate student noticing across a variety of mathematical (and nonmathematical) domains. This makes accessible a number of research questions associated with the underresearched construct of student noticing, such as: (a) Is what students initially notice in an instructional unit the most important factor in predicting how they will reason later? and (b) What does it take to support students noticing more productive features if an alternative center of focus has been prolonged? When the Noble, Nemirovsky, Dimattia, and Wright (2004) study was briefly described earlier in this article, readers may have questioned the importance of students not having noticed a second quantity of time. One might argue that the teacher could focus attention on this quantity later. However, it is also possible that students will be resistant to noticing a new quantity once a particular center of focus has been sustained for a period of time. These contrasting conjectures could be investigated empirically using a comparative cases design similar to this study and using the methodological tools developed here.

Furthermore, the focusing framework represents a contribution in its adaptation of transdisciplinary research on noticing to mathematics education. As a result, it extends the small body of previous research on mathematical noticing, in which the instructional environment was treated as a set of external forces acting upon students (Ellis & Grinstead, 2008; Lobato et al., 2003; Stevens & Hall, 1998), by better accounting for students' roles in the emergence of what is noticed. We explored student–student interactions (such as one student annotating another student's diagram in Class 1 to make a composed unit of distance and time visible), as well as student contributions to classroom discourse practices (such as the role of the student who initiated the shift back to noticing additive growth in Class 2). Therefore, development and refinement of the focusing framework should continue. Specifically, research is needed to explore relationships between various components of the framework (e.g., Will varying centers of focus emerge if the same tasks are used in two classrooms but the focusing interactions differ?). Applying the framework to other mathematical topics in other classroom settings with different ages of students may reveal missing components in the framework and identify additional types of focusing interactions (beyond highlighting, renaming, and quantitative dialogue).

Limitations and Future Research

One limitation of this study is that it does not account for why individuals deviated from group trends on the interview task. For example, we did not have an explanation for why one student from Class 1 engaged in nonmultiplicative reasoning. Such an account would likely require detailed information about students' pre-study mathematical understanding. This limitation could be tackled in future research by collecting additional data regarding students' prior knowledge.

Another constraint of our study was the use of small afterschool classes. However, we believe that within regular mathematics classrooms the same phenomena occur (i.e., students notice only some of the available information, and the instructional environments/interactions focus students' attention). Consequently, the challenge for extending our approach to classrooms of 35 students is methodological in nature. Specifically, in a large classroom, only a subset of students can feasibly be interviewed (due to the time-consuming nature of the accompanying qualitative data analysis), and these may not be the students who are most vocal in class (in terms of expressing what mathematical features they notice) or who make the most substantive contributions to the discourse practices that give rise to particular centers of focus. One way to address this challenge is to select a subset of target students to videotape during class and to interview; however, the results of the analysis of what these students notice would need to constitute a proxy measure for recording data on each individual student. Another potential limitation of research on students' noticing in regular mathematics classrooms is that the great pressures that teachers may feel to meet pacing guides and state standards could constrain the variety of mathematical foci that emerge.

Finally, this study was limited by the types of teacher data collected. As a result, a number of interesting questions remain unanswered regarding the teacher's role in the emergence of student noticing. For example, what is the relationship between what teachers plan to emphasize mathematically and the centers of focus that emerge during enacted lessons? How does what teachers notice during class influence what they help make visible for students? Do students' centers of focus always converge with what teachers see as important, and if not, what are the conditions for divergence? Addressing these questions in future research would require additional data regarding teachers' goals before each lesson is taught, more detailed information regarding teachers' mathematical understanding, and perhaps video-stimulated recall of teachers' interpretation of taught lessons.

Closing Remarks

By adopting the lens of student noticing, we have engaged in an investigation at a more macro level than is typical of the mathematics education research that examines student thinking. It might seem counterintuitive that the lens of noticing can illuminate aspects of thinking that a micro-level examination might not reveal. However, the data presented in this article demonstrate the potential value in looking systematically at what students notice and how noticing emerges in mathematics classrooms. We believe that a focus on student noticing will open up new possibilities for research and practice.

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Accepted on August 28, 2012